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Wave solution for an impulsively loaded rigid-plastic circular membrane

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Dedicated to Professor Piotr Perzyna on the occasion of his 70^{th} *birthday*

Transient response of a clamped rigid-perfectly plastic circular membrane subjected to central impulse loading is formulated as a wave propagation problem. A closed-form solution for transverse deflections is derived by neglecting the radial motion as well as the circumferential stress in the constitutive law but retaining finite deflections and slopes. The final shape of the membrane is obtained in terms of the magnitude of the applied impulse and the radius of the centrally loaded area.

Notations

ρ_0, ρ	=	initial and actual mass density per unit initial and current area of the membrane
R, r	=	initial and current radius
w	=	transverse displacement
t	=	time
p	=	pressure load per unit current area
σ_r, σ_θ	=	Cauchy stress components
σ_0	=	uniaxial yield stress
$\varepsilon_1, \varepsilon_2$	=	strain measures
σ_1, σ_2	=	stress measures conjugated to $\varepsilon_1, \varepsilon_2$
R_1	=	radius of central loaded area
R_0	=	radius of the plate
I_0	=	impulse per unit area
I	=	total impulse imparted to the plate
ξ	=	$\dfrac{R}{R_0}$, dimensionless radial coordinates
ξ_0	=	$\dfrac{R_1}{R_0}$, dimensionless central radius
τ	=	$\dfrac{c_0\, t}{R_0}$, dimensionless time

$c_0 = \sqrt{\dfrac{\sigma_o}{\rho_0}}$, plastic transverse wave velocity

$\omega = \dfrac{w}{R_0}$, dimensionless plate deflection

h = plate thickness

$s = \dfrac{\sigma_1}{\sigma_o}$, dimensionless stress

$v = \dfrac{\partial \omega}{\partial \tau}$

$u = \dfrac{\partial \omega}{\partial \xi}$

$V = \dfrac{I_0}{c_0 \rho_0 h}$, dimensionless impulse.

1. Introduction

Several decades of research on dynamic inelastic response of structures brought an important understanding of many factors that govern the deformation and failure of beams, plates, and shells. Perhaps the most comprehensive review of the methods and solutions pertaining to this subject was compiled by JONES [1]. Circular plates have been regarded as the prototype of a thin-walled structure on which various modeling concepts could be conveniently studied. Early work on plates was concerned with determining the transient and permanent deflection profile and relating it to material properties and temporal and spatial variation of the external dynamic loading applied in the form of a projectile impact, pressure loading or an ideal impulse [2, 3, 4, 5]. For a comprehensive review of the relevant literature, the reader is referred to the survey paper by JONES [6].

In a special level of complexity, failure of plates has become an important topic of research. It was shown through extensive testing that plates may fail either through necking followed by fracture (as in sheet metal forming) or through out-of-plane shear. JONES [7, 8] was first to offer a theoretical description of these phenomena, while NURICK and his co-workers contributed significantly to this problem through small-scale testing [9, 10, 11].

One of the present authors (T. W.) has been actively involved in the development of solution methods for dynamically loaded inelastic plates over more than 30 years. Early efforts were restricted to small deflection bending theory of viscoplastic plates subjected to a uniformly distributed impulsive loading (FLORENCE, WIERZBICKI [12]) and projectile impact (KELLY, WIERZBICKI [13]). These results have been extended to the range of moderately large deflection by WIERZBICKI and KELLY [14] and SYMONDS and WIERZBICKI [15], where the theoretical solution was correlated with tests. In a much more recent development, the momentum conservation approach was used to derive an approximate solution for large transient deformations of plates subjected to central explosive loading, WIERZBICKI and NURICK [16], and to mass impact, WIERZBICKI and HOO FATT [17]. It was shown by SYMONDS and WIERZBICKI [15] that large dynamic deformations of rigid-plastic plates subjected to an axisymmetric impulsive loading are governed by the homogeneous wave equation in the polar coordinate system. By contrast to the elastic formation, the initial-boundary value problem is subjected to an unloading condition that brings to the problem an interesting nonlinearity.

In previous attempts to treat this problem, analytically approximate solutions were derived. The mode solution with error minimization was developed in Ref. [15]. The method of eigenvalue expansion developed originally by WIERZBICKI [18] was applied in Ref. [16] while the momentum conservation approach was proposed in Ref. [17].

The objective of the present paper is to derive an exact solution of the problem using the method of characteristics. This method was very popular in the literature before the final element method came onto the scene in the seventies. Many important practical problems for inelastic solids and structures were solved using this method. A good source of information on this technique can be found in a classical book by CRISTESCU [19].

On the practical side, the present solution gives a distribution of maximum radial strains along the plate radius and the permanent deflection profile of the plate. Based on these results, predictions can be made on the onset of fracture as a function of the magnitude of the applied impulse and the radius of the centrally loaded area.

The authors believe that the subject of the paper nicely fits into this special anniversary volume of the Archives of Mechanics. The first author spent four months as a doctoral fellow back in the 60s in the Laboratory of Viscoplasticity directed by Professor Perzyna. The second author was the Ph.D. student of Professor Perzyna and worked closely with him during the period 1961 through 1981. By submitting this manuscript to the Archives of Mechanics, we would like to pay tribute to our wonderful teacher, mentor, and professional colleague. The present paper makes indeed a connection between the present time and Piotr's early work on the application of the newly developed by him theory of viscoplasticity. Back in 1963, after returning from his extended stay at Brown University, Piotr published two groundbreaking papers. The first of the series presented a unified, phenomenological theory of viscoplasticity (The constitutive equations for rate sensitive plastic materials, *Quarterly Applied Mathematics*, Vol. 20, pp. 321-332, 1963). The second dealt with propagation of spherical and cylindrical waves in the viscoplastic medium (On the propagation of stress waves in a rate sensitive plastic medium, *ZAMP*, Vol. 14, pp. 241-261, 1963). Our exposure to this subject, a mathematical rigor that has characterized all Piotr's work ever since and relation to the world of physics, made a long-lasting effect on our professional careers in these formative years. We take this opportunity to wish Piotr many happy and productive years and continuing success in his new world of fracture.

2. Formulation of the problem

The problem of the transverse motion of a rigid-perfectly plastic finite circular membrane is formulated and solved in this paper. We extend here the wave solution obtained in [20] for the transverse motion of a rigid-perfectly plastic string on a plastic foundation to the case of a circular membrane.

The circular membrane of a finite radius R_o is considered initially at rest on a plane and at $t = 0+$ an impulsive transverse load is applied over a central circular zone of radius R_1, Fig. 1.

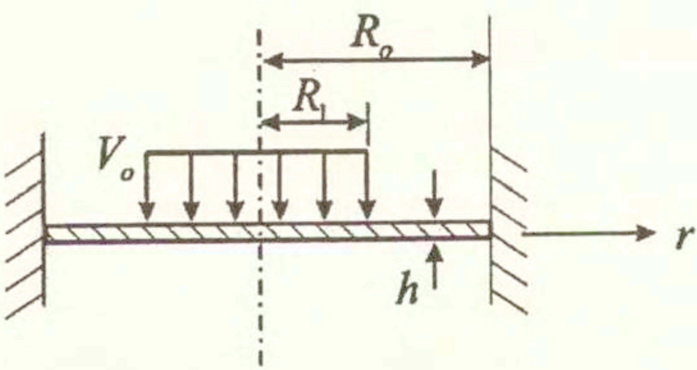

FIG. 1. Plate geometry and loading configuration.

In order to obtain a closed-form solution, it is assumed that the radial displacements are neglected. Also the circumferential stress is disregarded in the constitutive law. In order to simplify the problem, a *material* description with appropriate measures of stresses and strains is used in this work. A similar set of assumptions were also made in [17] to obtain a closed-form solution of a circular membrane impacted by a rigid projectile. In the present work, following a similar technique as in [20], the impulsive loading problem is transformed into a simpler but discontinuous initial-value problem. The piecewise smooth solution (i.e., the vertical displacement of the membrane) can then be constructed by using a complete analysis of the first and second order waves. The resulting permanent deflections depend on magnitude and spatial distribution of the applied impulse.

The stress profile is not always unique but this loss of uniqueness in stresses does not affect the strain, velocity, and displacement distribution in the membrane, which are unique. As an example, deflection and normalized deflection profiles are determined for several values of the radius of a centrally loaded area.

We consider a plane circular membrane clamped at the edge and subjected to the impact of a uniform transversal pressure p suddenly applied over a central part of the plate of radius $R_1 < R_o$. The pressure is held constant during a time interval $t_o > 0$ and then is suddenly removed. The problem is axi-symmetric and therefore the actual position of each particle of the membrane is completely described by its actual radius r and actual transversal displacement w,both r and w being functions of the initial radius R and the time t. We are interested in impulsive loading so we assume that for each fixed time interval t_o, the uniform applied pressure $p(t_o)$ is such that the product $t_o p(t_o)$ remains constant when t_o decreases, i.e., we have

$$\lim_{t_o \to 0} t_o p(t_o) = I_o = \text{const}, \qquad I_o = \rho_o V_o. \tag{2.1}$$

In the limiting case $t_o = 0$, the pressure loading problem is converted to the impulsive loading problem, i.e., an initial-boundary problem with a discontinuous initial velocity (see for instance [20] and also [21]).

The balance of momentum and the balance of mass in Lagrangean descriptions (according to MUNDAY and NEWITT [21], see also [19]) give rise to the following equations:

$$\rho_0 R \frac{\partial^2 r}{\partial t^2} = -\sigma_\theta \frac{ds}{dR} + \frac{\partial}{\partial R}\left(r\,\sigma_r \frac{dR}{ds}\frac{\partial r}{\partial R}\right),$$

$$\rho_0 R \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial R}\left(r\,\sigma_r \frac{dR}{ds}\frac{\partial w}{\partial R}\right), \tag{2.2}$$

$$\rho_r \frac{ds}{dR} = \rho_0 R,$$

where $r = r(R,t)$, $w = (R,t)$ represent the equation of the actual meridian curve of the membrane and

$$\mathrm{ds} = \left\{ \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial w}{\partial R}\right)^2 \right\}^{1/2} dR. \tag{2.3}$$

is the element of arc length of this curve; ρ_o and ρ are the initial and actual mass density per unit area, while σ_r and σ_θ are the actual meridian and circumferential (Cauchy) stress components respectively (on unit actual area). We use the following strain measures:

(2.4)
$$2\varepsilon_1 = \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial w}{\partial R}\right)^2 - 1,$$
$$2\varepsilon_2 = \frac{r^2}{R^2} - 1,$$

with the corresponding conjugated stress measures with respect to the mechanical power

(2.5)
$$\sigma_1 = \sigma_r \frac{r}{R}\left\{\left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial w}{\partial R}\right)^2\right\}^{-1/2},$$
$$\sigma_2 = \sigma_\theta \frac{R}{r}\left\{\left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial w}{\partial R}\right)^2\right\}^{1/2}.$$

Equation $((5.3)_2)$ then becomes

(2.6)
$$\rho_o R\frac{\partial^2 r}{\partial t^2} = \frac{\partial}{\partial R}\left(\sigma_1 R\frac{\partial r}{\partial R}\right) - \frac{r}{R}\sigma_2 .$$
$$\rho_o R\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial R}\left(\sigma_1 R\frac{\partial w}{\partial R}\right).$$
$$\rho\{(1+2\varepsilon_1)(1+2\varepsilon_2)\}^{1/2} = \rho_o .$$

The initial and boundary conditions are

(2.7)
$$(r,w)(R,0) = (R,0), \quad R \in (0,R_0),$$
$$\frac{\partial r}{\partial t}(R,0) = 0, \quad R \in (0,R_0),$$
$$\frac{\partial w}{\partial t}(R,0) = \begin{cases} -\dfrac{I_0}{\rho_0 h}, & R \in (0,R_1), \\ 0, & R \in (R_1,R_0), \end{cases}$$
$$w(R_0,t) = 0, t > 0.$$

The membrane is assumed to be rigid-perfectly plastic, with a uniaxial flow stress σ_o.

In order to get a closed-form solution of the above problem, two additional hypotheses are introduced: (i) there is no radial displacement, i.e., $r(R,t) \equiv R$

for $R \in (0, R_o)$ and $t > 0$; and, (ii) σ_2 is sufficiently small to be neglected in the constitutive relation. Both of these additional assumptions are very restrictive; the first one may be proper for clamped boundary conditions, while the second one is not adequate near the center of the membrane where by symmetry the circumferential stress is in fact equal to the meridian stress. Consequently, as it will be seen in the following, the solution of this simplified problem differs (at least in the neighborhood of the center of the membrane) from the experimentally observed one.

Under the assumptions (i) and (ii) the initial boundary value problem is simplified as follows:

$$\rho_o R \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial R}\left(\sigma_1 R \frac{\partial w}{\partial R}\right),$$

$$2\varepsilon_1 = \left(\frac{\partial w}{\partial R}\right)^2,$$

$$\frac{\partial \varepsilon_1}{\partial t} = \begin{cases} 0 \text{ for } \sigma_1 \in (0, \sigma_o) \text{ or } \sigma_1 = \sigma_o, \ \dfrac{\partial \sigma_1}{\partial t} < 0, \\ > 0 \text{ for } \sigma_1 = \sigma_o, \ \dfrac{\partial \sigma_1}{\partial t} = 0, \end{cases}$$

$$w(R, 0) = 0, R \in (0, R_o) \tag{2.8}$$

$$\frac{\partial w}{\partial t}(R, 0) = \begin{cases} -\dfrac{I_o}{\rho_o h}, & R \in (0, R_1) \\ 0, & \text{R} \in (R_1, R_o) \end{cases}$$

$$w(R_o, t) = 0, t > 0$$

where $\sigma_o = \text{const}$ is the uniaxial yield stress.

With the following notations:

$$\xi = \frac{R}{R_0}, \quad \xi_0 = \frac{R_1}{R_0}, \quad \tau = \frac{c_o t}{R_0}, \quad c_o = \left[\frac{\sigma_o}{\rho_o}\right]^{1/2}$$

$$\omega = \frac{w}{R_0}, \quad s = \frac{\sigma_1}{\sigma_o}, \quad V = \frac{I_o}{c_o \rho_o h} \tag{2.9}$$

$$u = \frac{\partial \omega}{\partial \xi}, \quad v = \frac{\partial \omega}{\partial \tau}.$$

a dimensionless form of problem (2.8) transformed to a system of first order PDE's is

$$
(2.10)\qquad
\begin{gathered}
\xi\frac{\partial v}{\partial \tau}=\frac{\partial}{\partial \xi}(\xi s u),\ \frac{\partial u}{\partial \tau}=\frac{\partial v}{\partial \xi}\\
\omega(\xi,0)=0,\xi\in(0,1)\\
v(\xi,0)=\begin{cases}-V,\ \xi\in(0,\xi_o)\\ 0,\ \xi\in(\xi_o,1),\end{cases}\\
\omega(1,\tau)=0,\tau>0.
\end{gathered}
$$

The natural boundary condition at the plate center requires vanishing of the total shear force, i.e., $\lim\limits_{\xi\to 0} 2\pi\xi s u=0$. This is satisfied as long as the product su remains finite as $\xi\to 0$. It can be shown by inspection that the present solution does satisfy the above boundary condition.

3. Solution of the problem

The initial discontinuity in velocity generates a discontinuous solution and therefore, in order to construct this solution, a complete analysis of all possible shock waves, rarefaction waves, and acceleration (second order) waves is necessary. This has been done in the earlier publication dealing with the impulsively-loaded string (see [20]). The main difference with the present problem being that rarefaction waves are possible in the membrane, at $\xi=0$.

The discontinuity in the initial condition$(10)_4$ gives rise at $(\xi_o,0)$ to two shock waves S_1 and S_2 with constant propagation speeds $d\xi/d\tau=1$ and $d\xi/d\tau=-1$ respectively (see Fig. 2).

No initial conditions for the stress have been prescribed but the initial values of s are of no consequence on the behavior of u and v for $\tau>0$ and therefore they need not to be given in the mathematical problem (2.10). Indeed, a horizontal shock wave at $\tau=0$ (which leaves u and v unchanged) will force s to jump at $(\xi_o,0)$ from its initial values to the value $s=1$ in order to make possible the propagation of S_1 and S_2. Furthermore, the only possible acceleration wave at $(\xi_o,0)$ is a vertical one (note that acceleration waves superposed on shock waves are not taken into consideration). Thus, the shock wave jump relations give the limit values, at $(\xi_o,0)$, of u and v beyond the two shocks, i.e.

$$
u=\frac{V}{2},\quad v=-\frac{V}{2}
$$

while the jump conditions for the first derivatives of u, v, and s give the limit values at $(\xi_o,0)$ of $\frac{\partial u}{\partial \tau}$ and $\frac{\partial s}{\partial \tau}$ beyond the two shocks, namely $\frac{\partial u}{\partial \tau}=\frac{\partial s}{\partial \tau}=0$.

Therefore, there is no way to decide whether beyond the shocks there is a rigid region (i.e., with $\frac{\partial \varepsilon}{\partial \tau} \equiv 0$) or a plastic region (i.e., with $\frac{\partial \varepsilon}{\partial \tau} > 0$).

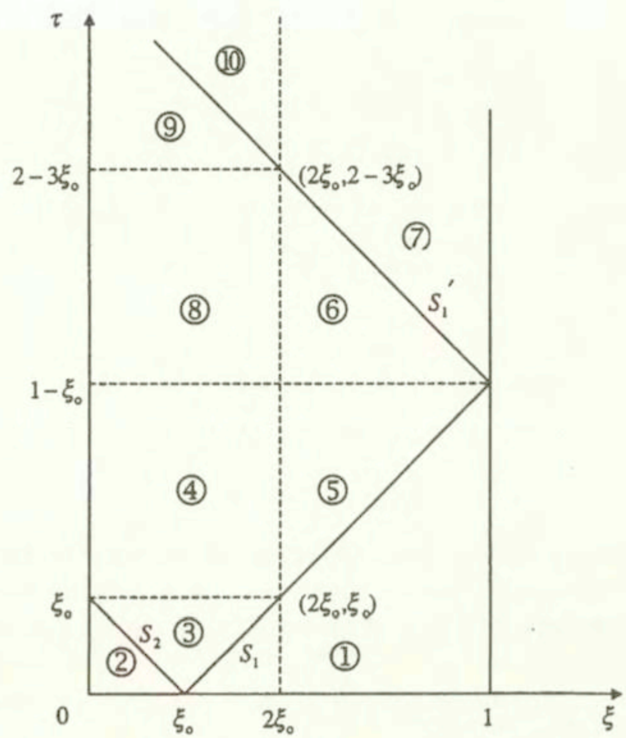

FIG. 2. A phase plane diagram showing converging (S_2) and diverging (S_1) wave initiated from ξ_o and the reflected wave (S_1').

Now, according to $(2.10)_{1-2}$ and the initial and boundary conditions, in the whole region ahead of S_1 the solution is $u \equiv v \equiv 0$, while in the whole region ahead of S_2 the solution is $u \equiv 0, v \equiv -V$. On the other hand, if a plastic region extends beyond the shocks, i.e., $s = 1$, $\frac{\partial \varepsilon}{\partial \tau} > 0$, a simple analysis of the solution in this case shows that the limit values of the strain ε on S_2 (beyond the shock) are rapidly increasing in time to infinity (at $\xi = 0$) while v is rapidly decreasing to $-\infty$. The conclusion is then that beyond the shocks S_1 and S_2 there has to be a rigid region with $\frac{\partial \varepsilon}{\partial \tau} = 0$ which implies $u(\xi, \tau) = u(\xi)$ and $v(\xi, \tau) = v(\tau)$ in that region.

From now on the technique employed to calculate u, v,and s in a rigid region is quite simple: equation $(2.10)_1$ is integrated with respect to ξ between the boundaries of the region (where $s = 1$) to get an ordinary differential equation for $v(\tau)$. Consequently $u(\xi,\tau) = u(\xi)$ can be calculated from its values on the boundary of the rigid region. In order to illustrate this technique we perform the explicit calculations for the first case.

We denote by "1" the region ahead of S_1, by "2" the region ahead of S_2, and by "3" the region beyond the two shocks. The solution in the first two regions can be readily obtained:

Region "1"

$$u(\xi,\tau) = v(\xi,\tau) = 0, \quad \tau \in (0, \min(1-\xi_o, \xi_o)),$$
$$\omega(\xi,\tau) = 0, \quad \xi \in (\xi_o + \tau, 1) \tag{3.1}$$

Region "2"

$$u(\xi,\tau) = 0 \quad \tau \in (0, \min(1-\xi_o, \xi_o)),$$
$$v(\xi,\tau) = -V \quad \xi \in (0, \xi_o - \tau), \tag{3.2}$$
$$\omega(\xi,\tau) = -V\tau.$$

It is interesting to observe that there is no unique solution for $s$in these two regions, but this has no influence on the values of u and v. Furthermore, in order to calculate the solution in region "3", one only needs the values of s on S_1 and S_2 (as s does not jump across these two shocks) which are equal to 1 for any choice of the stress solution in regions "1" and "2". Now, in order to calculate $v(\xi,\tau) = v(\tau)$ in region "3", Eq. (2.10) is integrated with respect to ξ between $\xi_1 = \xi_o - \tau$ and $\xi_2 = \xi_o + \tau$ (i.e., between S_2 and S_1) to give

$$\frac{1}{2}\{(\xi_o+\tau)^2 - (\xi_o-\tau)^2\}\frac{dv(\tau)}{d\tau} = (\xi_o+\tau)(su)(\xi_o+\tau,\tau) - (\xi_o-\tau)(su)(\xi_o-\tau,\tau) \tag{3.3}$$

where the values of u, v, and s are those of region "3". But $s(\xi_o+\tau,\tau) = s(\xi_o-\tau,\tau) = 1$, and from the jump relations across S_1 and S_2 we have

$$u_3(\xi_o+\tau,\tau) = -v_3(\xi_o+\tau,\tau),$$
$$u_3(\xi_o-\tau,\tau) = v_3(\xi_o-\tau,\tau) + V. \tag{3.4}$$

Introducing (3.4) into (3.3), the following differential equation is obtained for $v(t)$:

$$\tau\frac{dv}{d\tau} + v = V\frac{\tau - \xi_o}{2\xi_o}, \quad v(0) = -\frac{V}{2} \tag{3.5}$$

with the solution

$$v(\tau) = \frac{V(\tau - 2\xi_o)}{4\xi_o}.$$

The values of $u(\xi,\tau) = u(\xi)$ are then determined by means of (3.4) and (3.5)

$$u(\xi,\tau) = \frac{V(3\xi_o - \xi)}{4\xi_o}, \qquad \xi \in (\xi_o - \tau, \xi_o + \tau) \tag{3.6}$$

while the stress s is calculated by integrating Eq. (2.10) with respect to ξ from $\xi_1 = \xi_o - \tau$ to $\xi_2 = \xi$ (and $\xi_1 = \xi$ and $\xi_2 = \xi_o + \tau$ respectively) and using (3.5) and (3.6); one gets

$$s(\xi,\tau) = \frac{\xi^2 + 3(\xi_o^2 - \tau^2)}{2\xi(3\xi_o - \xi)}. \tag{3.7}$$

The expression of the deflection $\omega(\xi,\tau)$ in region "3" may be calculated either from u or from v since $u = \dfrac{\partial\omega}{\partial\xi}$ and $v = \dfrac{\partial\omega}{\partial\tau}$. The complete solution is then:

Region "3"

$$\begin{aligned}
u(\xi,\tau) &= \frac{V(3\xi_0 - \xi)}{4\xi_0},\\
v(\xi,\tau) &= \frac{V(\tau - 2\xi_0)}{4\xi_0},\\
s(\xi,\tau) &= \frac{\xi^2 + 3(\xi_0^2 - \tau^2)}{2\xi(3\xi_0 - \xi)},\\
\omega(\xi,\tau) &= \frac{V}{4}\left\{3(\xi - \xi_0) - \tau - \frac{1}{2\xi_0}\left(\xi^2 - (\xi_0 - \tau)^2\right)\right\}.
\end{aligned} \tag{3.8}$$

At this stage one has to treat separately the cases $\xi_o < 1 - \xi_o$ (i.e., $\xi_o < \frac{1}{2}$) and $\xi_o > 1 - \xi_o$ (i.e., $\xi_o > \frac{1}{2}$) as they depend on which one of the two shocks S_1 and S_2 is the first to reach the boundary $\xi = 1$ and the center $\xi = 0$, respectively. Of interest here is the final shape of the membrane after all its points have stopped, i.e., $\omega(\xi,\tilde{\tau})$ for $\tilde{\tau}$ the smallest τ with $v(\xi,\tilde{\tau}) = 0$ for all $\xi \in (0,1)$. Experiments show [6] that under impulsive transversal loading, the transversal velocity of the membrane $v(\xi,\tau)$ maintains a constant sign (which is negative in the present notation). Therefore transversal velocity and acceleration do vanish at the same time. In order to address the physical problem, the loading-unloading criterion must be introduced. It is assumed that whenever the instantaneous velocity vanishes at a given point $v(\xi,\tau)=0$, it must stay so for the duration of the motion. Then a rigid region will propagate over the membrane until the velocity vanishes at all points of the membrane; at that moment $\tilde{\tau}$ calculations are stopped, even if $\dfrac{\partial v}{\partial \tau} > 0$. The resulting deflections are called permanent deflection of the membrane.

$\mathbf{2}_1$ CASE $\xi_o < \dfrac{1}{2}$

The solution (3.8) for region "3" remains valid up to $\tau = \xi_o$ when S_2 reaches the center $\xi = 0$ of the membrane. Then, at $(0, \xi_o)$ a rarefaction fan appears in Region "3" for the stress s, with $s(\eta) = \dfrac{1}{\eta}$, $\eta = \dfrac{\xi}{\tau - \xi_o}$, $\eta \in (-\infty, -1)$, i.e., between S_2 and $\tau = \xi_o$; this rarefaction wave will "kill" the shock wave S_2 as it makes s decrease from the value $s = 1$ to the value $s = 0$. Indeed, we now have a Goursat problem for $(2.10)_{1-2}$ at $(0, \xi_o)$, between $\tau = \xi_o$ (i.e., $\eta = +\infty$) and $\xi = 0$ (i.e., $\eta = 0$), with $(su)(\eta = 0) = 0$ from the boundary condition at $\xi = 0$ and $(u, v, s)(\eta = +\infty) = (\dfrac{3V}{4}, -\dfrac{V}{4}, 0)$ as the limit values calculated from (3.8); but u can no longer be equal to zero at the center $\xi = 0$ since $u(\eta = +\infty) = \dfrac{3V}{4} > 0$ and the constitutive law does not allow ε to decrease, so $s(\eta = 0)$ has to vanish. Therefore S_2 can not be reflected at $(0, \xi_o)$ as there exists no wave mechanism which allows $s(\eta)$ to increase from $s = 0$ at $\eta = +\infty$ to the value $s = 1$ at $\eta = 1$ (and thus to permit a reflected shock wave to propagate) and subsequently decrease from $s = 1$ at $\eta = 1$ to $s = 0$ at $\eta = 0$. The only wave that starts propagating at $(0, \xi_o)$ is a horizontal acceleration wave $\tau = \xi_o$ which has to change the sign of $\dfrac{\partial s}{\partial \tau}$ since $\dfrac{\partial s}{\partial \tau}(\xi, \xi_o)$ given by (3.8) is negative in the neighborhood of $\xi = 0$ while $\xi \to 0$ $s(\xi, \xi_o) = 0$. This horizontal second order wave gives rise to a vertical acceleration wave at $(2\xi_o, \xi_o)$ where it meets the shock S_1. So for $\tau \in (\xi_o, 1 - \xi_o)$ there are two new regions: "4" and "5" (see Fig. 1), both of them being rigid regions and, following the same technique as that employed to calculate the solution in region "3", one gets

Region "4"

$$
\begin{aligned}
& u(\xi,\tau) = \frac{V(3\xi_0 - \xi)}{4\xi_0}, \\
(3.9) \qquad & v(\xi,\tau) = -\frac{V\xi_0^2}{(\xi_0 + \tau)^2}, \qquad \xi \in (0, 2\xi_0), \quad \tau \in (\xi_0, 1 - \xi_0), \\
& s(\xi,\tau) = \frac{4\xi_0^3 \xi}{(3\xi_0 - \xi)(\xi_0 + \tau)^3},
\end{aligned}
$$

(3.9) [cont.]
$$\omega(\xi,\tau)=\frac{V\,\xi_0^2}{\xi_0+\tau}-\frac{V}{8\,\xi_0}\left(3\,\xi_0^2+(\xi-3\,\xi_0\,)^2\right),$$

Region "5"

(3.10)
$$\begin{aligned}
u(\xi,\tau)&=\frac{V\,\xi_o^2}{\xi^2},\\
v(\xi,\tau)&=-\frac{V\,\xi_o^2}{(\xi_o+\tau\,)^2}\qquad \tau\in(\xi_o,1-\xi_o)\quad \xi\in(2\,\xi_o,\xi_o+\tau),\\
v(\xi,\tau)&=-\frac{V\,\xi_o^2}{(\xi_o+\tau\,)^2},\\
s(\xi,\tau)&=\frac{\xi^3}{(\xi_o+\tau\,)^3},\\
\omega(\xi,\tau)&=V\,\xi_o^2\left(\frac{1}{\xi_o+\tau}-\frac{1}{\xi}\right).
\end{aligned}$$

Now the shock wave S_1 reaches the edge of the membrane at $\tau=1-\xi_o$ and is reflected as the shock wave S_1' while a horizontal acceleration wave $\tau=1-\xi_o$ is also generated at $(1,1-\xi_o)$ in order to change the sign of $\dfrac{\partial s}{\partial\tau}$ and make $\dfrac{\partial s}{\partial\tau}$ increase again in time for $\tau>1-\xi_o$, in order to allow the propagation of S_1'. The limit value of the derivatives at $(1,-\xi_o)$ in region "7" (see Fig. 1) is

$$\frac{\partial u}{\partial\tau}=-V\,\xi_o^2\left(1+\frac{\partial s}{\partial\tau}\right)$$

and therefore $\dfrac{\partial u}{\partial\tau}=0$, $\dfrac{\partial s}{\partial\tau}=-1$ and region "7" is again a rigid region with $v(\xi,\tau)=v(\tau)=0$ since $v=0$ at the edge $\xi=1$. One can proceed further by calculating the solution in regions "6" and "8" and then in region "7". This gives, respectively

Region "6"

$$u(\xi,\tau) = \frac{V\xi_o^2}{\xi^2},$$

$$v(\xi,\tau) = V\xi_o^2\left\{\frac{1}{(2-\xi_o-\tau)^2} - 2\right\}$$

(3.11) $$\tau \in (1-\xi_o, 2-3\xi_o) \quad \tau \in (2\xi_o, 2-\xi_o-\tau),$$

$$s(\xi,\tau) = \frac{\xi^3}{\xi_o^2(2-\xi_o-\tau)^3},$$

$$\omega(\xi,\tau) = V\xi_o^2\left\{\frac{1}{2-\xi_o-\tau} - \frac{1}{\xi} + 2(1-\xi_o-\tau)\right\}.$$

Region "8"

(3.12)

$$u(\xi,\tau) = \frac{V(3\xi_o-\xi)}{4\xi_o},$$

$$v(\xi,\tau) = V\xi_0^2\left\{\frac{1}{(2-\xi_o-\tau)^2} - 2\right\} \quad \rho \in (0, 2\xi_o) \quad \tau \in (1-\xi_o, 2-3\xi_o),$$

$$s(\xi,\tau) = \frac{4\xi_o^3\xi}{(3\xi_o-\xi)(2-\xi_o-\tau)^3},$$

$$\omega(\xi,\tau) = V\xi_0^2\left\{\frac{1}{2-\xi_o-\tau} + 2(1-\xi_o-\tau)\right\} - \frac{V}{8\xi_o}\left\{3\xi_o^2 + (\xi-3\xi_o)^2\right\}.$$

Region "7"

$$u(\xi,\tau) = 2V\xi_o^2 \quad \tau \in (1-\xi_o, 2-3\xi_o) \quad \xi \in (2-\xi_o-\tau, 1),$$

(3.13) $$v(\xi,\tau) = 0 \quad \text{and} \quad \omega(\xi,\tau) = 2V\xi_o^2(\xi-1),$$

$$\tau \in (2-3\xi_o, 2-\xi_o) \quad \xi \in (2\xi_o, 1),$$

It can be noted that v may now vanish in regions "6" and "8" so two subcases should be considered.

2.1a For $\xi_o \in (0, \frac{1}{2\sqrt{2}})$ it is found that $v(\xi,\tilde{\tau}) = 0$ in regions "6" and "8", with $\tilde{\tau} = 2 - \xi_o - \frac{1}{\sqrt{2}} < 2 - 3\,\xi_o$ and the final shape of the membrane is given by

$$\omega\,(\xi, \tilde{\tau}) = \begin{cases} 2(\sqrt{2}-1)V\,\xi_o^2 - \frac{V}{8\,\xi_o}(3\,\xi_o^2 + (\xi - 3\,\xi_o\,)^2), \xi \in (0, 2\,\xi_o) \\ V\,\xi_o^2\left\{2(\sqrt{2}-1) - \frac{1}{\xi}\right\}, \xi \in (2\,\xi_o, \frac{1}{\sqrt{2}}) \\ 2V\,\xi_o^2(\xi - 1), \xi \in (\frac{1}{\sqrt{2}}, 1) \end{cases} \tag{3.14}$$

2.1b For $\xi_o \in (\frac{1}{2\sqrt{2}}, \frac{1}{2})$, v remains negative in regions "6" and "8" up to $\tau = 2 - 3\,\xi_o$ so one has to calculate the solution in regions "9" and "10"

Region "9"

$$u(\xi, \tau) = \frac{V(3\xi_o - \xi)}{4\xi_o},$$

$$v(\xi, \tau) = \frac{V}{2\xi_o}\left(2 - \tau - \frac{5}{2}\xi_o - 4\xi_o^3 - 3\xi_o \ln\frac{2 - \tau - \xi_o}{2\xi_o}\right),$$

$$\begin{aligned} \omega(\xi, \tau) = &-\frac{V}{8\xi_o}(\xi - 3\xi_o)^2 \\ &+\frac{V}{2\xi_o}\left(-3\xi_o(\tau - 2 + \xi_o)\ln\frac{2 - \xi_o - \tau}{2\xi_o} - \frac{\tau^2}{2} - \tau(4\xi_o^3 - \frac{\xi_o}{2} - 2)\right. \\ &\left. -4\xi_o^4 + 4\xi_o^3 + \frac{25}{4}\xi_o^2 - \xi_o - 2\right), \end{aligned} \tag{3.15}$$

$$\tau \in (2 - 3\xi_o, 2 - \xi_0), \quad \xi \in (0, 2 - \xi_o - \tau).$$

Region "10"

$$\begin{aligned} u(\xi, \tau) &= \frac{V}{4\,\xi_o}\left\{8\,\xi_o^3 + 6\,\xi_o - 3\xi + 6\,\xi_o \ln\frac{\xi}{2\,\xi_o}\right\} \\ v(\xi, \tau) &= 0 \\ \omega(\xi, \tau) &= \frac{V}{8\,\xi_o}\left\{3\,\xi^2 - 16\,\xi_o^3\right\} + 16\,\xi_o^3 - 12\,\xi_o^2 - 12\,\xi_o\,\xi \ln\frac{\xi}{2\,\xi_o} \end{aligned} \tag{3.16}$$

$$\tau \in (2 - 3\xi_o,\, 2 - \xi_o), \quad \xi \in (2 - \xi_o - \tau, 2\,\xi_o)$$

The expression of v in region "9" implies that there exists a unique $\tilde{\tau} \in (2 - 3\xi_o,\ 2 - \xi_o)$ such that $v(\xi, \tilde{\tau}) = 0$ in region "9" and $\tilde{\tau}$ is the solution of the algebraic equation.

$$(3.17) \qquad 2\mu - 3\ln\mu = \frac{3}{2} + 4\xi_o^2, \ \ \mu = \frac{2 - \tilde{\tau} - \xi_o}{2\xi_o} \in (0, 1).$$

The motion of the membrane does therefore stop before S_1' reaches the center $\xi = 0$ and the final shape of the membrane is given by $\omega(\xi, \tilde{\tau})$ calculated from $(3.16)_3$ for region "10" and $(3.15)_3$, $(3.13)_3$ for regions "9" and "7" respectively, i.e.

$$(3.18) \quad \omega(\xi, \tilde{\tau}) = \begin{cases} -\dfrac{V}{8\xi_0}(\xi - 3\xi_0)^2 \\ \qquad + \dfrac{V}{2\xi_0}\left(\dfrac{\tilde{\tau}^2}{2} + 2(2\xi_0 - 1\tilde{\tau} - 4\xi_0^3 + \dfrac{35}{4}\xi^2 - 8\xi_0 + 2\right), \quad \xi \in (0, \dfrac{1}{\sqrt{2}}) \\ -\dfrac{V}{8\xi_0}\left(3\xi^2 - 16\xi_0^3\xi + 16\xi_0^3 - 12\xi_0^2 - 12\xi_0\xi\ln\dfrac{\xi}{2\xi_0}\right), \quad \xi \in (\dfrac{1}{\sqrt{2}}, 2\xi_0) \\ 2V\xi_o^2(\xi - 1), \quad \xi \in (2\xi_0, 1) \end{cases}$$

2.2 CASE $\xi_o = \dfrac{1}{2}$

In this case, for the regions "1", "2", and "3" (see Fig. 3), the solution is the same as for the case $\xi_0 < \dfrac{1}{2}$, i.e., it is given by (3.1), (3.2), and (3.8) respectively where $\xi_0 = \dfrac{1}{2}$.

Region "4"'

$$u(\xi, \tau) = \frac{V(3 - 2\xi)}{4}, \qquad \tau \in (\frac{1}{2}, \frac{3}{2}),$$

$$(3.19) \qquad v(\xi, \tau) = V\left(\frac{1}{4} - \tau - \frac{3}{2}\ln\frac{3 - 2\tau}{2}\right), \quad \rho \in (0, \frac{3}{2} - \tau),$$

$$\omega(\xi, \tau) = \frac{V}{4}\left(-\xi^2 + 3\xi - 2\tau^2 + 7\tau - 5 - 3(2\tau - 3)\ln\frac{3 - 2\tau}{2}\right).$$

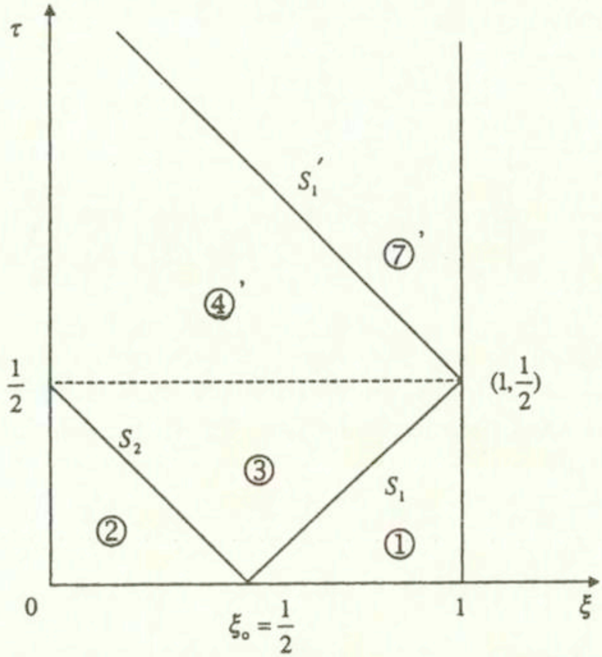

FIG. 3. A special case of the wave solution shown in Fig. 2 in which $\xi_o = 1/2$.

Region "7'"

$$u(\xi,\tau) = V\left(2 - \frac{3}{2}\xi + \frac{3}{2}\ln\xi\right),$$

(3.20)
$$v(\xi,\tau) = 0, \quad \tau \in (\frac{1}{2},\frac{3}{2}) \quad \xi \in (\frac{3}{2}-\tau, 1),$$

$$\omega(\xi,\tau) = \frac{V}{4}\left(-3\,\xi^2 + 2\xi + 1 + 6\xi\ln\xi\right).$$

There exists a unique value $\tilde{\tau} \in (\frac{1}{2},\frac{3}{2})$, i.e. before the shock wave S_1' reaches the center $\xi = 0$, such that $v(\xi,\tilde{\tau})$ in Region "4", namely $\tilde{\tau}$, is the solution of the algebraic equation

(3.21)
$$\frac{3}{2}\ln\frac{3-2\tilde{\tau}}{2} + \tilde{\tau} = \frac{1}{4}.$$

The membrane stops moving at $\tau = \tilde{\tau}$ and its final shape is given by $(3.19)_3$ and $(3.20)_3$, that is

$$
(3.22)\quad \omega(\xi,\tilde{\tau}) = \begin{cases} \dfrac{V}{4}\left(-\xi^2+3\xi-2\tilde{\tau}^2+7\tilde{\tau}-5-3(2\tilde{\tau}-3)\ln\dfrac{3-2\tilde{\tau}}{2}\right), \\ \hfill \xi\in\left(0,\dfrac{3}{2}-\tilde{\tau}\right), \\ \dfrac{V}{4}\left(-3\xi^2+2\xi+1+6\xi\ln\xi\right),\ \ \xi\in\left(\dfrac{3}{2}-\tilde{\tau}1\right). \end{cases}
$$

2.3 CASE $\xi_o = 1$

There is only one initially generated shock wave in this case, namely S_2, starting at the edge of the membrane (see Fig. 4); the solution in region "2" is given by (3.2) and

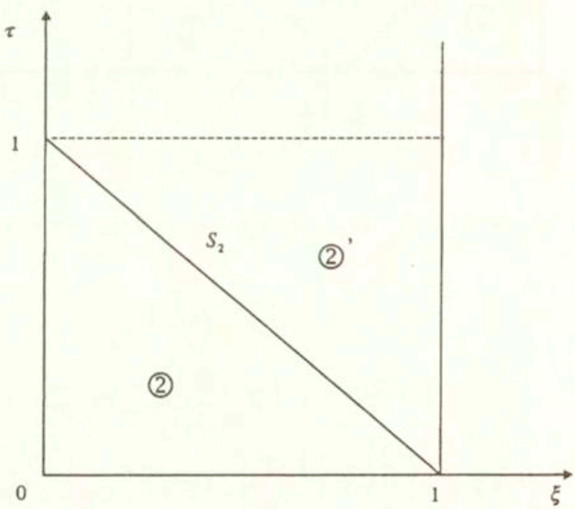

FIG. 4. Degenerated wave picture on the phase plane in the case of $\xi_o = 1$ showing one converging wave.

Region "2"

$$
(3.23)\qquad \begin{aligned} u(\xi,\tau) &= V, \\ v(\xi,\tau) &= 0, & \tau\in(0,1), \\ s(\xi,\tau) &= \frac{1-\tau}{\xi}, & \xi\in(1-\tau,1), \\ \omega(\xi,\tau) &= V(\xi-1). \end{aligned}
$$

The motion stops at $\tilde{\tau} = 1$ and the final shape of the membrane is therefore given by

$$\omega(\xi, 1) = V(\xi - 1), \quad \xi \in (0, 1), \tag{3.24}$$

4. Discussion

Despite its apparent formal character, the present solution brings a wealth of interesting information about transient response of thin plates. According to the present wave approach, the deflected shape of the plate depends strongly on the value of the dimensional radius of loading $\xi_o = R_0/R$. There are three regions of ξ_o and in each of them a different set of equations describes the deflected shape. In the region $\xi_o \in \left(0, 1/2\sqrt{2}\right)$, Eq. (3.14) applies. Then the region $\xi_o \in \left(1/2\sqrt{2}, 1/2\right)$ is governed by Eq. (3.16). Finally, the solution for $\xi_0 = 1$ is given by Eq. (3.24). A comparison of deflected shapes for five different values of the parameter ξ_o is shown in Fig. 5. It is seen that the smaller the dimensional radius of impulsive

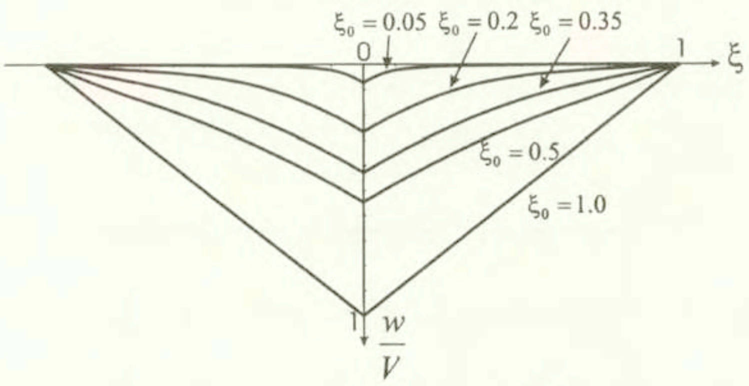

FIG. 5. Dimensionless deflection profiles of the membrane for different values of the radius of the impulsive loading.

loading ξ_o is, the more localized deformations are around the plate center. It is interesting to plot the dimensionless maximum central amplitude and $(w/V)_{\max}$ versus the value of the parameter ξ_o(Fig. 6).

It is seen that the above relationship is almost linear except for the initial slightly curved portion. As mentioned earlier, the nonlinearity of the problem comes not from the wave solution but rather from the unloading condition. For the rigid-perfectly plastic material, unloading occurs whenever a velocity of the given particle of the beam becomes zero. The equation of the unloading wave is given by Eq. (3.17) which is a nonlinear algebraic equation. However, when the

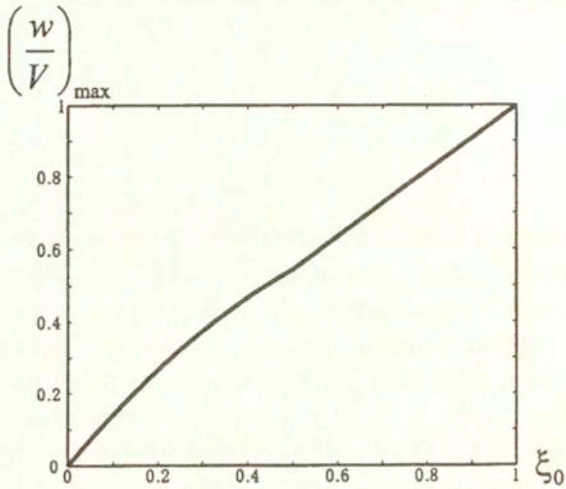

FIG. 6. A plot of the dimensionless maximum central deflection of the plate as a function of the radius of the impassive loading.

unloading time $\bar{\tau}$ is plotted against the parameter ξ_o, the unloading boundary is seen to be composed of two portions of almost straight lines as shown in Fig. 7.

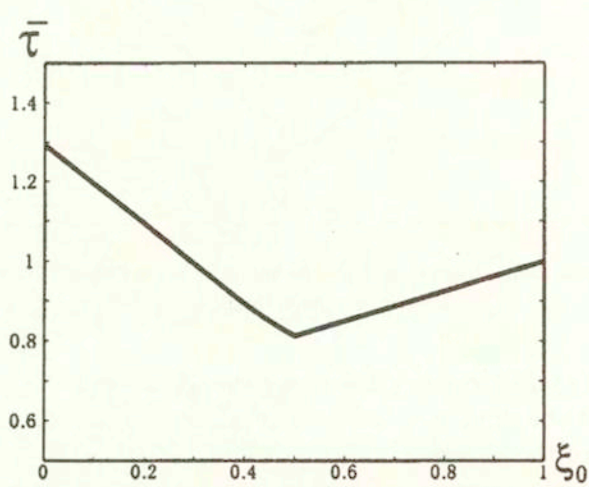

FIG. 7. The dependence of the so-called "time-to-rest" on the radius of the impulsive loading.

So far considered was the case when the amplitude of the initial velocity V was independent of the radius of the impulsive loading. It is interesting to rearrange the solution and assume that the total impulse imparted to the plate, defined by

$$V_{\text{total}} = \pi\xi^2 V \tag{4.1}$$

is held constant. Under this condition, the dimensionless maximum central amplitude is no longer an increasing function as it was in the case shown in Fig. 6 but is a decreasing function of the parameter ξ_o (see Fig. 8).

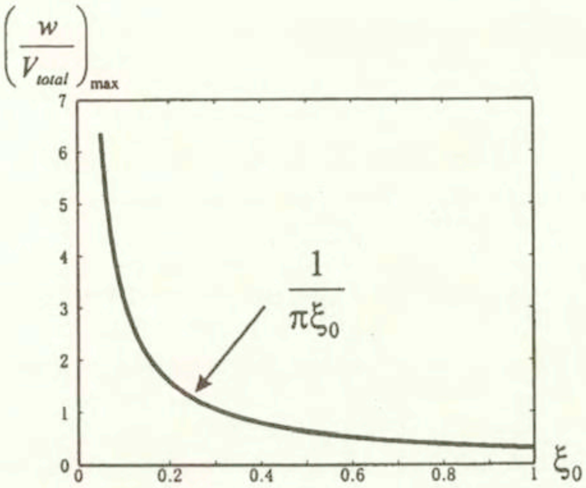

FIG. 8. The dependence of dimensionless maximum central deflection on the parameter ξ_o for a constant value of the total applied impulse V_{total}.

Finally, it should be noted that the maximum slope of the deflected shape is always constant in the range $\xi_o \in (0, 1/2)$ and is equal to $\omega' = 3V/4$. Then in the range $\xi_o \in (1/2, 1)$ the slope will increase and assume a maximum value at $\xi_o = 1$. In the theory of thin membranes the radial strain is defined by $\varepsilon = 1/2\,(w')^2$. With this definition and using the calculated slope, it is possible to make an estimation on the maximum strain developed in the membrane as a result of the impulsive loading. The membrane will fracture when the strain reaches a critical value ε_f, that is when

$$V_{\text{cric}} = \sqrt{\frac{32}{9}\varepsilon_f} \tag{4.2}$$

where V_{cric} denotes critical velocity to fracture.

In conclusion, it must be stated that the present analysis provides the first closed form solution of the problem of impulsively loaded thin plates loaded by an explosive material distributed around the center part of the plate. TEELING-SMITH and NURICK [11] performed a series of tests of impulsively loaded thin plates and determined experimentally the deflected shapes. A comparison of the present theory with those tests will be done in a future publication.

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