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Localisation effect during wave pulse propagation in randomly stratified elastic medium

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IN THIS PAPER the propagation of the planar wave pulses in a two-dimensional randomly stratified elastic medium is considered. The waves propagate in the plane (x_1, x_2) and are independent of the third variable z. The problem is described by means of the transition matrix method. The transition matrix and the wave equation for a homogeneous layer is presented. The equations for the wave fields reflected from and transmitted through the randomly stratified elastic slab are derived. The theoretical results are illustrated by graphically presented numerical calculations. A possibility of application of the wave analysis to modelling of the fatigue crack propagation initiated at the interfaces and continued due to sequences of the reflected and transmitted stress waves is discussed.

1. Introduction

IN THE PAPER we consider wave pulses in two-dimensional elastic stratified medium. The results represent a certain generalisation of those given in [3], where the wave pulses in a one-dimensional elastic medium were considered. On the other hand, the paper extends the model presented in [4] for two-dimensional harmonic waves to the non-stationary phenomenon of wave pulses. The main mathematical tool used for the analysis of the problem is the so-called *transition matrix method*. This method is very effective in solving wave problems in stratified media – both deterministic and stochastic. The historical development of the transition matrix approach was presented in [5] and our other papers on wave problems in the stochastic stratified media [3, 4]. The advantage of the method is the fact that one can perform large part of solving the equation procedure analytically and only the last inversion of the Fourier transformation of the wave amplitudes must be numerical.

The schedule of the paper is the following. First we introduce the notation used throughout the paper and present the derivation of the Fourier transformed wave equation for wave pulses in a homogeneous layer. Then we derive the

equation for the planar elastic wave pulse and give the expressions for its solution. In Sec. 3 we consider waves in the medium built of homogeneous parallel layers. We introduce such a system of independent variables that interfaces of the layers (where the material parameters change their values) are perpendicular to the x-axis. Inside of each layer the wave field satisfies the wave equations derived in Sec. 2 and at the discontinuity planes the displacements and traction vector are continuous. Finally we derive the equation for the (Fourier transformed) wave field in all the layered medium and solve it, obtaining the analytical expressions for the amplitudes of the reflected and transmitted wave pulses. In Sec. 4 we present the results of an example calculating numerically the inverse Fourier transforms and observing the evolution in time of the pulses. Section 5 gives a discussion of the possible effect of the observed phenomena on the fatigue crack propagation initiated at the interfaces and continued due to sequences of the reflected and transmitted stress waves.

2. The wave equation for a single layer

We consider a linear elastic wave propagating in the homogeneous isotropic medium. The wave propagation is governed by the following system of partial differential equations [7]:

$$\rho \frac{\partial^2}{\partial t^2} u_i = \sigma_{ij,j}, \qquad i = 1, 2, 3, \tag{2.1}$$

where σ_{ij} is the stress tensor defined as

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} \tag{2.2}$$

(double indexes denote the summation from 1 to 3; the subscript "1" corresponds to the independent variable x, "2" to y and "3" to z). In the above equation λ and μ are the elastic Lamé constants and ρ is density of the medium.

We consider waves propagating in the plane (x, y) and independent of the third spatial variable z. In our co-ordinate system the wave field has got the following form:

$$\mathbf{u}(x, y, z, t) = (u_1(x, y, t), u_2(x, y, t), 0)^T. \tag{2.3}$$

Condition (2.3) makes some elements of the stress tensor equal to zero and slightly simplifies the governing equations. Since in further considerations we deal with the problem of wave propagation in a layered medium, where the interfaces of homogeneous layers are perpendicular to the x-axis, we can equivalently describe the wave problem (2.1) – (2.2) by the following matrix differential equation

$$\frac{d}{dx}\hat{\mathbf{u}} = \mathbf{M}\hat{\mathbf{u}}, \tag{2.4}$$

with the solution $\hat{\mathbf{u}}$ searched for and the system matrix $\mathbf{M}$ defined as:

$$\mathbf{M} = \begin{bmatrix} 0 & -ik\alpha & \kappa & 0 \\ -ik & 0 & 0 & \eta \\ -\omega^2\rho & 0 & 0 & -ik \\ 0 & k^2\beta - \omega^2\rho & -ik\alpha & 0 \end{bmatrix}, \qquad \hat{\mathbf{u}} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{\tau}_1 \\ \hat{\tau}_2 \end{bmatrix}, \tag{2.5}$$

where $\hat{u}_1$, $\hat{u}_2$, $\hat{\tau}_1$ and $\hat{\tau}_2$ are the Fourier transform of the non-zero displacement co-ordinates, u_1, u_2, and the non-zero co-ordinates of the traction vector, τ_1 and τ_2; the parameters in the matrix are (see [5]):

$$\alpha = \frac{\lambda}{(\lambda + 2\mu)}, \qquad \beta = \frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)}, \qquad \kappa = \frac{1}{(\lambda + 2\mu)}, \qquad \eta = \frac{1}{\mu}.$$

In solving the wave problem described by equation (2.4) for a single layer, a boundary condition:

$$\hat{\mathbf{u}}(0, k, \omega) = \hat{\mathbf{u}}_0(k, \omega) = \begin{bmatrix} \hat{u}_1(0, k, \omega) \\ \hat{u}_2(0, k, \omega) \\ \hat{\tau}_1(0, k, \omega) \\ \hat{\tau}_2(0, k, \omega) \end{bmatrix}, \tag{2.6}$$

representing jointly the incident wave pulse reaching the plane $x = 0$ and the pulse reflected from it, has to be introduced. Then the value at the opposite side of the layer, at $x = L$, say, can be represented as:

$$\hat{\mathbf{u}}(L, k, \omega) = \mathbf{T}(L)\hat{\mathbf{u}}_0(k, \omega), \tag{2.7}$$

where $\mathbf{T}(L)$ is the transition matrix for the two-dimensional waves propagating through the layer of thickness L. This matrix can be represented by the following Sylvester formula [11, 5]:

$$\mathbf{T}(L) = \exp\{\mathbf{M}L\} = \sum_{i=1}^{4} \left(\mathbf{\Pi} \frac{(\mathbf{M} - p_k \mathbf{Id})}{p_i - p_k} \right) \exp\{p_i, L\} \tag{2.8}$$

where $p_i, i = 1, 2, 3, 4$ are the eigenvalues of the system matrix $\mathbf{M}$:

$$p_{1,2} = \pm \frac{\sqrt{k^2\mu - \omega^2\rho}}{\sqrt{\mu}}, \qquad p_{3,4} = \pm \frac{\sqrt{k^2(\lambda + 2\mu) - \omega^2\rho}}{\sqrt{\lambda + 2\mu}}. \tag{2.9}$$

The elements of the transition matrix have got a rather complicated form. They are presented explicitly in [5].

The transition matrix $\mathbf{T}(.)$ makes it possible to express the wave field $\hat{\mathbf{u}}$,

$$\hat{\mathbf{u}}(x,k,\omega) = \begin{bmatrix} \hat{u}_1(x,k,\omega) \\ \hat{u}_2(x,k,\omega) \\ \hat{\tau}_1(x,k,\omega) \\ \hat{\tau}_2(x,k,\omega) \end{bmatrix}, \tag{2.10}$$

at any point $x \in [0,L] \subset R^+$ in a homogeneous medium (inside a layer), provided the boundary condition $\hat{\mathbf{u}}_0 = \hat{\mathbf{u}}(0,k,\omega)$ at $x = 0$ is known. This wave field has got the form (2.7) with $\mathbf{T}(x)$ as the transition matrix.

3. Elastic waves in the layered medium

The approach developed in Sec. 2 allows us also to describe the transition of the two-dimensional elastic wave through a multi-layered medium. In such a case, knowing the transition matrices through any individual layers, we can obtain the transition matrix through all the stratified medium as a product of them.

Let us consider the multi-layered medium (slab) built of N layers of elastic materials, with thickness $\Delta_j, j = 1,2,,\ldots,N$. Assume that the stratified medium is surrounded by the homogeneous elastic environment, at $x < 0$ and $x > L = \sum_{j=1}^{N} \Delta_j$. Since the wave field $\hat{\mathbf{u}}$ must be continuous at the layer interfaces, the wave on the back surface, at $x = L$, can be expressed in the following form:

$$\hat{\mathbf{u}}(L) = \mathbf{T}_N(\Delta_N)\mathbf{T}_{N-1}(\Delta_{N-1})...\mathbf{T}_2(\Delta_2)\mathbf{T}_1(\Delta_1)\hat{\mathbf{u}}_0, \tag{3.1}$$

where $\hat{\mathbf{u}}_0$ is the boundary condition at $x = 0$, $\hat{\mathbf{u}}(L)$ is the vector of the transmitted wave and $\mathbf{T}_j(.)$, for $j = 1,2,...,N$, is the transition matrix for the j-th layer, depending on the material parameters (possibly random).

In Equation (3.1) all the material properties of the multi-layered medium are completely described by a 4×4 matrix $\mathbf{T}$, being the product of the transition matrices through the individual layers and interpreted as a transition matrix through the slab built of N layers of homogeneous elastic materials:

$$\mathbf{T} = \prod_{j=1}^{N} \mathbf{T}_j(\Delta_j) = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix}. \tag{3.2}$$

Let us notice that the vector $\hat{\mathbf{u}}_0$ describes jointly the (Fourier transforms of) incident wave pulse (going to the right) and all the reflected pulses leaving the slab (going to the left), generated by all the reflections at the interfaces of the layers, measured at the plane $x \equiv 0$. Analogously, $\hat{\mathbf{u}}(L)$ represents the transmitted pulses measured at the plane $x \equiv L$ and generated by all reflections at the internal interfaces of the layers and transmitted through the layers. To make the obtained formulae effective we must separate the incident and reflected waves from $\hat{\mathbf{u}}_0$. The waves in every region of environment surrounding the layered slab can be described in the following way. In the left-hand environment (for $x_1 < 0$) there is the incident (right-going) wave, represented as:

$$\begin{bmatrix} \hat{u}_1^{\mathrm{inc}}(x) \\ \hat{u}_2^{\mathrm{inc}}(x) \\ \hat{\tau}_1^{\mathrm{inc}}(x) \\ \hat{\tau}_2^{\mathrm{inc}}(x) \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \exp\{-ip_1 x\} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \exp\{-ip_3 x\}, \tag{3.3}$$

and the reflected (left-going) wave:

$$\begin{bmatrix} \hat{u}_1^{\mathrm{ref}}(x) \\ \hat{u}_2^{\mathrm{ref}}(x) \\ \hat{\tau}_1^{\mathrm{ref}}(x) \\ \hat{\tau}_2^{\mathrm{ref}}(x) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \exp\{ip_1 x\} + \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} \exp\{ip_3 x\}. \tag{3.4}$$

In the right-hand environment (for $x > L$) there is only the transmitted (right-going) wave, having the following form:

$$\begin{bmatrix} \hat{u}_1^{\mathrm{tr}}(x) \\ \hat{u}_2^{\mathrm{tr}}(x) \\ \hat{\tau}_1^{\mathrm{tr}}(x) \\ \hat{\tau}_2^{\mathrm{tr}}(x) \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} \exp\{-ip_1 x\} + \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \exp\{-ip_3 x\}. \tag{3.5}$$

Since the displacements and tractions are related quantities, the number of independent amplitude constants, $A_1 - F_4$, can be reduced. Calculating the Fourier transforms of the respective Eqs. (2.2) we obtain:

$$\hat{\tau}_1 = (\lambda + 2\mu)\frac{d\hat{u}_1}{dx} + ik\hat{u}_2 \tag{3.6}$$

and

$$\hat{\tau}_2 = i\mu k\hat{u}_1 + \mu\frac{d\hat{u}_2}{dx}, \tag{3.7}$$

what applied in (3.3) – (3.5), results in the following expressions for the incident, reflected and transmitted wave:

$$
(3.8)\qquad \begin{bmatrix} \hat{u}_1^{\mathrm{inc}}(x) \\ \hat{u}_2^{\mathrm{inc}}(x) \\ \hat{\tau}_1^{\mathrm{inc}}(x) \\ \hat{\tau}_2^{\mathrm{inc}}(x) \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ i[-(\lambda+2\mu)p_1A_1 + kA_2] \\ i\mu(kA_1 - p_1A_2) \end{bmatrix} \exp\{-ip_1x\} + \begin{bmatrix} B_1 \\ B_2 \\ i[-(\lambda+2\mu)p_3B_1 + kB_2] \\ i\mu(kB_1 - p_3B_2) \end{bmatrix} \exp\{-ip_3x\},
$$

$$
(3.9)\qquad \begin{bmatrix} \hat{u}_1^{\mathrm{ref}}(x) \\ \hat{u}_2^{\mathrm{ref}}(x) \\ \hat{\tau}_1^{\mathrm{ref}}(x) \\ \hat{\tau}_2^{\mathrm{ref}}(x) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ i[(\lambda+2\mu)p_1C_1 + kC_2] \\ i\mu(kC_1 + p_1C_2) \end{bmatrix} \exp\{ip_1x\} + \begin{bmatrix} D_1 \\ D_2 \\ i[(\lambda+2\mu)p_3D_1 + kD_2] \\ i\mu(kD_1 + p_3D_2) \end{bmatrix} \exp\{ip_3x\}.
$$

$$
(3.10)\qquad \begin{bmatrix} \hat{u}_1^{\mathrm{tr}}(x) \\ \hat{u}_2^{\mathrm{tr}}(x) \\ \hat{\tau}_1^{\mathrm{tr}}(x) \\ \hat{\tau}_2^{\mathrm{tr}}(x) \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ i[-(\lambda+2\mu)p_1E_1 + kE_2] \\ i\mu(kE_1 - p_1E_2) \end{bmatrix} \exp\{-ip_1x\} + \begin{bmatrix} F_1 \\ F_2 \\ i[-(\lambda+2\mu)p_3F_1 + kF_2] \\ i\mu(kF_1 - p_3F_2) \end{bmatrix} \exp\{-ip_3x\}.
$$

In these equations the parameters of the incident wave, A_1, A_2, B_1, B_2, are calculated from the postulated incident wave pulse. The remaining parameters are included into the wave Eq. (3.2) where we substitute $x = 0$ and $x = L$ into the left and right-hand waves, respectively. Then we have got the system of 4

algebraic equations for 8 unknown parameters, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2. To solve them uniquely we need 4 additional conditions on the wave amplitudes. The conditions are concerned with the directions of motion of the wave pulses (reflected and transmitted) governed by the direction of the incident pulse. The relations are analogous to the Snelius law known in optics or in the theory of harmonic elastic wave propagation.

As usually, the derivation of the reflection law is based on the assumption that the component of the wave vector parallel to the reflection (transmission) surface must be continuous. Let us denote the wave vector by $\mathbf{p}$. Then the law can be written as

$$\mathbf{p}_{\parallel}^{\mathrm{inc}} = \mathbf{p}_{\parallel}^{\mathrm{ref}} = \mathbf{p}_{\parallel}^{\mathrm{tr}}, \tag{3.11}$$

or equivalently:

$$\mathbf{p}^{\mathrm{inc}} \sin\varphi^{\mathrm{inc}} = \mathbf{p}^{\mathrm{ref}} \sin\varphi^{\mathrm{ref}} = \mathbf{p}^{\mathrm{tr}} \sin\varphi^{\mathrm{tr}}, \tag{3.12}$$

where φ^{inc}, φ^{ref} and φ^{tr} are, respectively, the angles of incidence, reflection and transmission of waves of certain type (longitudinal and transversal).

From the formulae (3.8) – (3.10) we see that the Fourier-transformed waves propagate only in the direction perpendicular to the interface surface. We also know the co-ordinate of the wave vectors perpendicular to the interface surface; it is $\mathbf{p}_{\perp} = p_3$ for the longitudinal wave and $\mathbf{p}_{\perp} = p_1$ for the transversal one. Knowing the incidence angle we can reduce the number of unknown constants in (3.8) – (3.10), using the Snelius-like relations (3.12).

Assume that the incident wave pulse is only longitudinal one and it reaches the interface surface at the angle of incidence $\varphi^{\mathrm{inc}} = \alpha$ (Fig. 1). This means that the amplitudes can be represented as

$$A_1 = A_2 = 0, \qquad B_1 = B\cos\alpha, \qquad B_2 = B\sin\alpha. \tag{3.13}$$

Hence the incident wave takes the following form:

$$\begin{bmatrix} \hat{u}_1^{\mathrm{inc}}(x) \\ \hat{u}_2^{\mathrm{inc}}(x) \\ \hat{\tau}_1^{\mathrm{inc}}(x) \\ \hat{\tau}_2^{\mathrm{inc}}(x) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ i[-(\lambda+2\mu)p_3 B_1 + kB_2] \\ i\mu(kB_1 - p_3 B_2) \end{bmatrix} \exp\{-ip_3 x\} \tag{3.14}$$

$$= \begin{bmatrix} B\cos\alpha \\ B\sin\alpha \\ iB[-(\lambda+2\mu)p_3\cos\alpha + k\sin\alpha] \\ i\mu B(k\cos\alpha - p_3\sin\alpha) \end{bmatrix} \exp\{-ip_3 x\}.$$

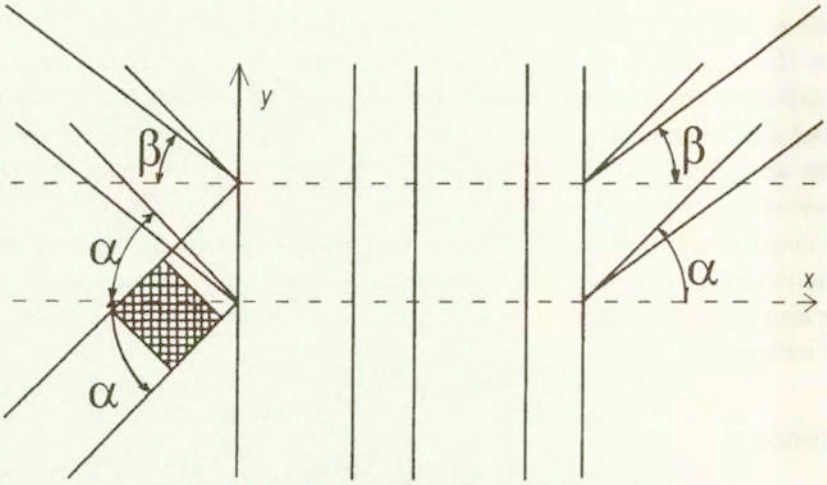

FIG. 1. The model of layered medium ($n = 2$ pairs of layers).

To obtain the relations restricting the number of amplitude constants, we transform the formula (3.12) into the more convenient form

$$\mathbf{p}_{\parallel}^{\mathrm{inc}} = \mathbf{p}_{\perp}^{\mathrm{inc}} \tan \varphi^{\mathrm{inc}} = \mathbf{p}_{\parallel}^{\mathrm{ref}} = \mathbf{p}_{\perp}^{\mathrm{ref}} \tan \varphi^{\mathrm{ref}} = \mathbf{p}_{\parallel}^{\mathrm{tr}} = \mathbf{p}_{\perp}^{\mathrm{tr}} \tan \varphi^{\mathrm{tr}}. \tag{3.15}$$

The solution of amplitude problem requires the calculation of the following angles:

$\varphi_{\mathrm{lon}}^{\mathrm{ref}}$ – the angle for reflected longitudinal wave;

$\varphi_{\mathrm{tr}}^{\mathrm{ref}}$ – the angle for reflected transversal wave;

$\varphi_{\mathrm{lon}}^{\mathrm{tr}}$ – the angle for transmitted longitudinal wave;

$\varphi_{\mathrm{tr}}^{\mathrm{tr}}$ – the angle for transmitted transversal wave.

Similarly to the case of harmonic waves [3], the above defined angles satisfy the following conditions:

$$\varphi_{\mathrm{lon}}^{\mathrm{ref}} = \varphi_{\mathrm{lon}}^{\mathrm{tr}} = \varphi_{\mathrm{lon}}^{\mathrm{inc}} = \alpha \tag{3.16}$$

and

$$\varphi_{\mathrm{tr}}^{\mathrm{ref}} = \varphi_{\mathrm{tr}}^{\mathrm{tr}} = \beta \tag{3.17}$$

Using formula (3.15) we find

$$p_3 \tan \varphi_{\mathrm{lon}}^{\mathrm{inc}} = p_1 \tan \varphi_{\mathrm{tr}}^{\mathrm{ref}} \tag{3.18}$$

or eventually

$$\beta = \arctan\left(\frac{p_3}{p_1} \tan \alpha\right) \tag{3.19}$$

Knowing the angles, we can write the direction vectors of every wave in the form:

$\mathbf{n} = (\cos\alpha, \sin\alpha)$ – the direction vector for incident longitudinal wave (the same as for transmitted longitudinal wave);

$\mathbf{m} = (-\cos\alpha, \sin\alpha)$ – the direction vector for reflected longitudinal wave;

$\mathbf{g} = (\cos\beta, \sin\beta)$ – the direction vector for transmitted transversal wave;

$\mathbf{h} = (-\cos\beta, \sin\beta)$ – the direction vector for reflected transversal wave.

Since the displacements of the medium for transversal waves are perpendicular to the direction vector, the displacement vectors are ($\mathbf{a}$ is the unit vector perpendicular to the reference plane):

$\mathbf{a} \times \mathbf{g} = (-\sin\beta, \cos\beta)$ – the displacement vector for transmitted transversal wave;

$\mathbf{a} \times \mathbf{h} = (-\sin\beta, -\cos\beta)$ – the displacement vector for reflected transversal wave.

The displacement vectors for the longitudinal waves are the same as those for the direction vectors.

Now, using the vectors defined above and the relations analogous to (3.13), we transform the waves (3.9) and (3.10) to the following form:

$$\begin{bmatrix} \hat{u}_1^{\mathrm{ref}}(x) \\ \hat{u}_2^{\mathrm{ref}}(x) \\ \hat{\tau}_1^{\mathrm{ref}}(x) \\ \hat{\tau}_2^{\mathrm{ref}}(x) \end{bmatrix} = -C \begin{bmatrix} \sin\beta \\ \cos\beta \\ i[(\lambda+2\mu)p_1 \sin\beta + k\cos\beta] \\ i\mu(k\sin\beta + p_1\cos\beta) \end{bmatrix} \exp\{ip_1 x\} + D \begin{bmatrix} -\cos\alpha \\ \sin\alpha \\ i[-(\lambda+2\mu)p_3\cos\alpha + k\sin\alpha] \\ i\mu(-k\cos\alpha + p_3\sin\alpha) \end{bmatrix} \exp\{ip_3 x\}. \tag{3.20}$$

$$\begin{bmatrix} \hat{u}_1^{\mathrm{tr}}(x) \\ \hat{u}_2^{\mathrm{tr}}(x) \\ \hat{\tau}_1^{\mathrm{tr}}(x) \\ \hat{\tau}_2^{\mathrm{tr}}(x) \end{bmatrix} = E \begin{bmatrix} -\sin\beta \\ \cos\beta \\ i[(\lambda+2\mu)p_1 \sin\beta + k\cos\beta] \\ i\mu(-k\sin\beta - p_1\cos\beta) \end{bmatrix} \exp\{-ip_1 x\} + F \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ i[-(\lambda+2\mu)p_3\cos\alpha + k\sin\alpha] \\ i\mu(k\cos\alpha - p_3\sin\alpha) \end{bmatrix} \exp\{-ip_3 x\}. \tag{3.21}$$

To complete the wave Eq. (3.1) we conclude that

$$(3.22)\qquad \hat{\mathbf{u}}_0 = \begin{bmatrix} \hat{u}_1^{\mathrm{inc}}(0) \\ \hat{u}_2^{\mathrm{inc}}(0) \\ \hat{\tau}_1^{\mathrm{inc}}(0) \\ \hat{\tau}_2^{\mathrm{inc}}(0) \end{bmatrix} + \begin{bmatrix} \hat{u}_1^{\mathrm{ref}}(0) \\ \hat{u}_2^{\mathrm{ref}}(0) \\ \hat{\tau}_1^{\mathrm{ref}}(0) \\ \hat{\tau}_2^{\mathrm{ref}}(0) \end{bmatrix} = B \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ i[-(\lambda+2\mu)p_3\cos\alpha + k\sin\alpha] \\ i\mu(k\cos\alpha - p_3\sin\alpha) \end{bmatrix}$$

$$-C \begin{bmatrix} \sin\beta \\ \cos\beta \\ i[(\lambda+2\mu)p_1\sin\beta + k\cos\beta] \\ i\mu(k\sin\beta + p_1\cos\beta) \end{bmatrix} + D \begin{bmatrix} -\cos\alpha \\ sin\alpha \\ i[-(\lambda+2\mu)p_3\cos\alpha + k\sin\alpha] \\ i\mu(-k\cos\alpha + p_3\sin\alpha) \end{bmatrix}$$

and

$$(3.23)\qquad \hat{\mathbf{u}}(L) = \begin{bmatrix} \hat{u}_1^{\mathrm{tr}}(L) \\ \hat{u}_2^{\mathrm{tr}}(L) \\ \hat{\tau}_1^{\mathrm{tr}}(L) \\ \hat{\tau}_2^{\mathrm{tr}}(L) \end{bmatrix} = E \begin{bmatrix} -\sin\beta \\ \cos\beta \\ i[(\lambda+2\mu)p_1\sin\beta + k\cos\beta] \\ i\mu(-k\sin\beta - p_1\cos\beta) \end{bmatrix} \exp\{-ip_1L\}$$

$$+F \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ i[-(\lambda+2\mu)p_3\cos\alpha + k\sin\alpha] \\ i\mu(k\cos\alpha - p_3\sin\alpha) \end{bmatrix} \exp\{-ip_3L\}.$$

The aim of calculations is to obtain the parameters C, D, E and F from the Eq. (3.1) if the amplitude B of the incident wave is known.

In the particular case, if the incidence angle $\alpha = 0$, we have got:

$$(3.24)\qquad \hat{\mathbf{u}}_0 = B \begin{bmatrix} 1 \\ 0 \\ -i(\lambda+2\mu)p_3 \\ i\mu k \end{bmatrix} - C \begin{bmatrix} 0 \\ 1 \\ ik \\ i\mu p_1 \end{bmatrix} - D \begin{bmatrix} 1 \\ 0 \\ i(\lambda+2\mu)p_3 \\ i\mu k \end{bmatrix}$$

and

$$\hat{\mathbf{u}}(L) = E \begin{bmatrix} 0 \\ 1 \\ ik \\ -i\mu p_1 \end{bmatrix} \exp\{-ip_1 L\} \tag{3.25}$$

$$+F \begin{bmatrix} 1 \\ 0 \\ -i(\lambda+2\mu)p_3 \\ i\mu k \end{bmatrix} \exp\{-ip_3 L\}.$$

To proceed with calculations tending to the solution of the algebraic wave Eqs. (3.1) – (3.2), we substitute (3.24) – (3.25) and obtain:

$$\begin{bmatrix} F\exp\{-ip_3 L\} \\ E\exp\{-ip_1 L\} \\ ikE\exp\{-ip_1 L\} - i(\lambda+2\mu)p_3 F\exp\{-ip_3 L\} \\ -i\mu p_1 E\exp\{-ip_1 L\} + i\mu k F\exp\{-ip_3 L\} \end{bmatrix} \tag{3.26}$$

$$+ \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} D \\ C \\ ikC + i(\lambda+2\mu)p_3 D \\ i\mu p_1 C + i\mu k D \end{bmatrix}$$

$$= \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} B \\ 0 \\ -i(\lambda+2\mu)p_3 B \\ i\mu k B \end{bmatrix}.$$

Ordering the terms in (3.26) we write the system of equations in the following form:

$$
(3.27)\qquad
\begin{bmatrix}
T_{12}+ikT_{13}+i\mu p_1 T_{14} & T_{11}+i(\lambda+2\mu)p_3T_{13}+i\mu kT_{14} \\
T_{22}+ikT_{23}+i\mu p_1 T_{24} & T_{21}+i(\lambda+2\mu)p_3T_{23}+i\mu kT_{24} \\
T_{32}+ikT_{33}+i\mu p_1 T_{34} & T_{31}+i(\lambda+2\mu)p_3T_{33}+i\mu kT_{34} \\
T_{42}+ikT_{43}+i\mu p_1 T_{44} & T_{41}+i(\lambda+2\mu)p_3T_{43}+i\mu kT_{44}
\end{bmatrix}
\begin{bmatrix}
0 & \exp\{-ip_3L\} \\
\exp\{-ip_1L\} & 0 \\
ik\exp\{-ip_1L\} & -i(\lambda+2\mu)p_3\exp\{-ip_3L\} \\
i\mu p_1\exp\{-ip_1L\} & i\mu k\{-ip_3L\}
\end{bmatrix}
\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix}
$$

$$
= \begin{bmatrix}
T_{11}-i(\lambda+2\mu)p_3T_{13}+i\mu kT_{14} \\
T_{21}-i(\lambda+2\mu)p_3T_{23}+i\mu kT_{24} \\
T_{31}-i(\lambda+2\mu)p_3T_{33}+i\mu kT_{34} \\
T_{41}-i(\lambda+2\mu)p_3T_{43}+i\mu kT_{44}
\end{bmatrix} B.
$$

4. Numerical results

In Sec. 3 we presented the analytical formulae solving the two-dimensional dynamical wave problem in layered medium. However, to make the proposed method fully effective, we have to proceed with computer calculations. The computational procedure leads from the initial wave pulse that travels through the homogeneous half-space and reaches the front interface of the stratified layer, to the pulses that are generated by the multiple reflections within the stratified slab. There result two pulses (with a very complicated structure): the reflected one that goes back in the homogeneous half-space and the transmitted one that propagates further beyond the back surface of the stratified medium.

The equations applied for calculations, that is Eq. (3.1) and the following ones, are written for the quantities (matrices and vectors) depending on two parameters k and ω (the variables of Fourier transformation with respect to the spatial variable and the time). Solving Eq. (3.1) we do this for a fixed pair of variables k and ω. A complete solution of the problem requires the Eq. (3.1) to be solved in the whole domain of the variables. Thereafter, the inverse Fourier transformation allows us to determine the evolution of the pulses in the actual time.

The scheme of calculations for some incident wave pulse can be given as follows. Let us assume, for simplicity, that the incident wave pulse is longitudinal

and reaches the front surface of the stratified layer at the angle of incidence $\alpha = 0$. In this case the Fourier transformation of the excitation, c.f. Eq. (3.14), takes the following form:

$$\begin{bmatrix} \hat{u}_1^{\mathrm{inc}}(x,k,\omega) \\ \hat{u}_2^{\mathrm{inc}}(x,k,\omega) \\ \hat{\tau}_1^{\mathrm{inc}}(x,k,\omega) \\ \hat{\tau}_2^{\mathrm{inc}}(x,k,\omega) \end{bmatrix} = B \begin{bmatrix} 1 \\ 0 \\ -i(\lambda+2\mu)p_3 \\ i\mu k \end{bmatrix} \exp\{-ip_3 x\}. \tag{4.1}$$

We have to remember that the eigenvalues p_1 and p_3 defined in (2.9) are functions of variables k and ω, as the coefficient B must have the form:

$$B = B(k,\omega), \qquad k,\omega \in (-\infty,\infty). \tag{4.2}$$

Moreover, the elements of the transition matrix $\mathbf{T}$ defined in (3.2) depend also on the variables k and ω, i.e.:

$$\mathbf{T} = \mathbf{T}(k,\omega). \tag{4.3}$$

To solve the problem, we substitute the calculated coefficient B in equation (3.27) and solve it with respect to the coefficients C, D, E and F. All of them are some functions of k and ω. We substitute the calculated values of the coefficients in the expressions (3.20) and (3.21) defining the reflected and transmitted wave pulses (or, more precisely, their Fourier transformations). Those expressions for $\alpha = 0$ take the following form:

$$\begin{bmatrix} \hat{u}_1^{\mathrm{ref}}(0,k,\omega) \\ \hat{u}_2^{\mathrm{ref}}(0,k,\omega) \\ \hat{\tau}_1^{\mathrm{ref}}(0,k,\omega) \\ \hat{\tau}_2^{\mathrm{ref}}(0,k,\omega) \end{bmatrix} = -C \begin{bmatrix} 0 \\ 1 \\ ik \\ i\mu p_1 \end{bmatrix} - D \begin{bmatrix} 1 \\ 0 \\ i(\lambda+2\mu)p_3 \\ i\mu k \end{bmatrix}, \tag{4.4}$$

$$\begin{bmatrix} \hat{u}_1^{\mathrm{tr}}(L,k,\omega) \\ \hat{u}_2^{\mathrm{tr}}(L,k,\omega) \\ \hat{\tau}_1^{\mathrm{tr}}(L,k,\omega) \\ \hat{\tau}_2^{\mathrm{tr}}(L,k,\omega) \end{bmatrix} = E \begin{bmatrix} 0 \\ 1 \\ ik \\ -i\mu p_1 \end{bmatrix} \exp\{-ip_1 L\} + F \begin{bmatrix} 1 \\ 0 \\ -i(\lambda+2\mu)p_3 \\ i\mu k \end{bmatrix} \exp\{-ip_3 L\}. \tag{4.5}$$

Two-dimensional inverse Fourier transformation of the vectors (4.4) and (4.5) give us eventually the pulses as the functions of time, t, and the second space variable, y, i.e.:

$$(4.6)\qquad \begin{bmatrix} u_1^{\mathrm{ref}} \\ u_2^{\mathrm{ref}} \\ \tau_1^{\mathrm{ref}} \\ \tau_2^{\mathrm{ref}} \end{bmatrix} = \begin{bmatrix} u_1^{\mathrm{ref}}(0,y,t) \\ u_2^{\mathrm{ref}}(0,y,t) \\ \tau_1^{\mathrm{ref}}(0,y,t) \\ \tau_2^{\mathrm{ref}}(0,y,t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_1^{\mathrm{tr}} \\ u_2^{\mathrm{tr}} \\ \tau_1^{\mathrm{tr}} \\ \tau_2^{\mathrm{tr}} \end{bmatrix} = \begin{bmatrix} u_1^{\mathrm{tr}}(L,y,t) \\ u_2^{\mathrm{tr}}(L,y,t) \\ \tau_1^{\mathrm{tr}}(L,y,t) \\ \tau_2^{\mathrm{tr}}(L,y,t) \end{bmatrix}.$$

It should be noticed that practical calculations must be performed in several steps. First we must choose a grid in the region of changes of the variables k and ω, and then express the coefficient $B(k,\omega)$ over this grid. Next we solve the Eq. (3.27) for all points of this grid to obtain the values of coefficients C, D, E and F in all grid points. Finally, we have got the expressions (4.4) and (4.5) in the grid points and we can calculate the inverse Fourier transformation that is the solution of the problem.

As an example, in numerical calculation we have considered a slab built of two metals: steel and titanium surrounded by an aluminium environment. Material constants for aluminium, steel and titanium are, respectively, cf. [12]:

$$\lambda^0 = 5.44 \times 10^{10}\ \mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{sec}^{-2}, \quad \mu^0 = 2.75 \times 10^{10}\ \mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{sec}^{-2},$$

$$\lambda^1 = 10.71 \times 10^{10}\ \mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{sec}^{-2}, \quad \mu^1 = 8.14 \times 10^{10}\ \mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{sec}^{-2},$$

$$\lambda^2 = 7.08 \times 10^{10}\ \mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{sec}^{-2}, \quad \mu^2 = 4.31 \times 10^{10}\ \mathrm{kg}\cdot\mathrm{m}^{-1}\cdot\mathrm{sec}^{-2},$$

$$\rho^0 = 2750\ \mathrm{kg}\cdot\mathrm{m}^{-3},$$

$$\rho^1 = 8670\ \mathrm{kg}\cdot\mathrm{m}^{-3},$$

$$\rho^2 = 4300\ \mathrm{kg}\cdot\mathrm{m}^{-3}.$$

Such a material configuration was chosen because the values of corresponding parameters differ from each other and enable us to observe the strong effect of stratification on the wave. We consider the incident longitudinal wave pulse of a constant unit amplitude and a finite duration of reaching the stratified slab of the fixed thickness at some finite instant (see Fig. 2). In our calculations the time axis is scaled in seconds while the spatial variable is represented by numbers of points of the grid used in calculations (we used 128 points of the spatial grid and 4096 points of the grid over time).

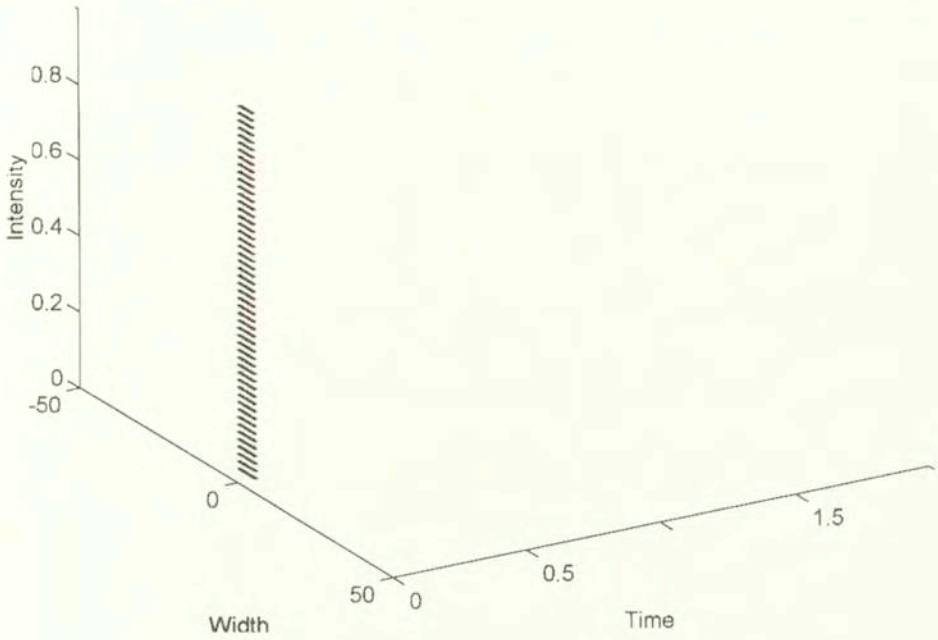

FIG. 2. Incident longitudinal wave pulse.

In our numerical studies we assumed that the material of the slab is located in pairs of the same thickness: one layer of steel and one layer of titanium. At Figure 3 we presented the evolution of the wave pulse longitudinal displacement (measured at the back surface of the slab) transmitted through the slab built of only one pair of layers steel-titanium, each of thickness $L/2$. We observe that the first wave pulse reaches the back surface after some time (we call it the travel time through the slab) and then it is followed by a number of pulses generated by the left and right-going pulses due to multiple reflections and transmissions at all the interfaces (internal interfaces of the layers within the slab and the interfaces of the slab and the surrounding environment). We see that the maximal peaks (except for the first peak generated directly by the transmitted incident pulse) correspond to the pulses reflected from the internal interfaces of the layers (the distance between peaks is the doubled travel time through a single layer). If the number of pairs of layers in the slab grows, we can observe an interesting effect of homogenisation. Figure 4 shows that for $n = 20$ pairs of layers (each layer of thickness $L/40$), the distances between maximal peaks are equal (approximately) to the double travel time through all the slab. Figure 5 shows the maps of peaks for the growing number of pairs of layers in the slab (from $n = 1$ to $n = 20$). We can see how this effect of homogenisation increases together with stronger mixing of the material in the slab.

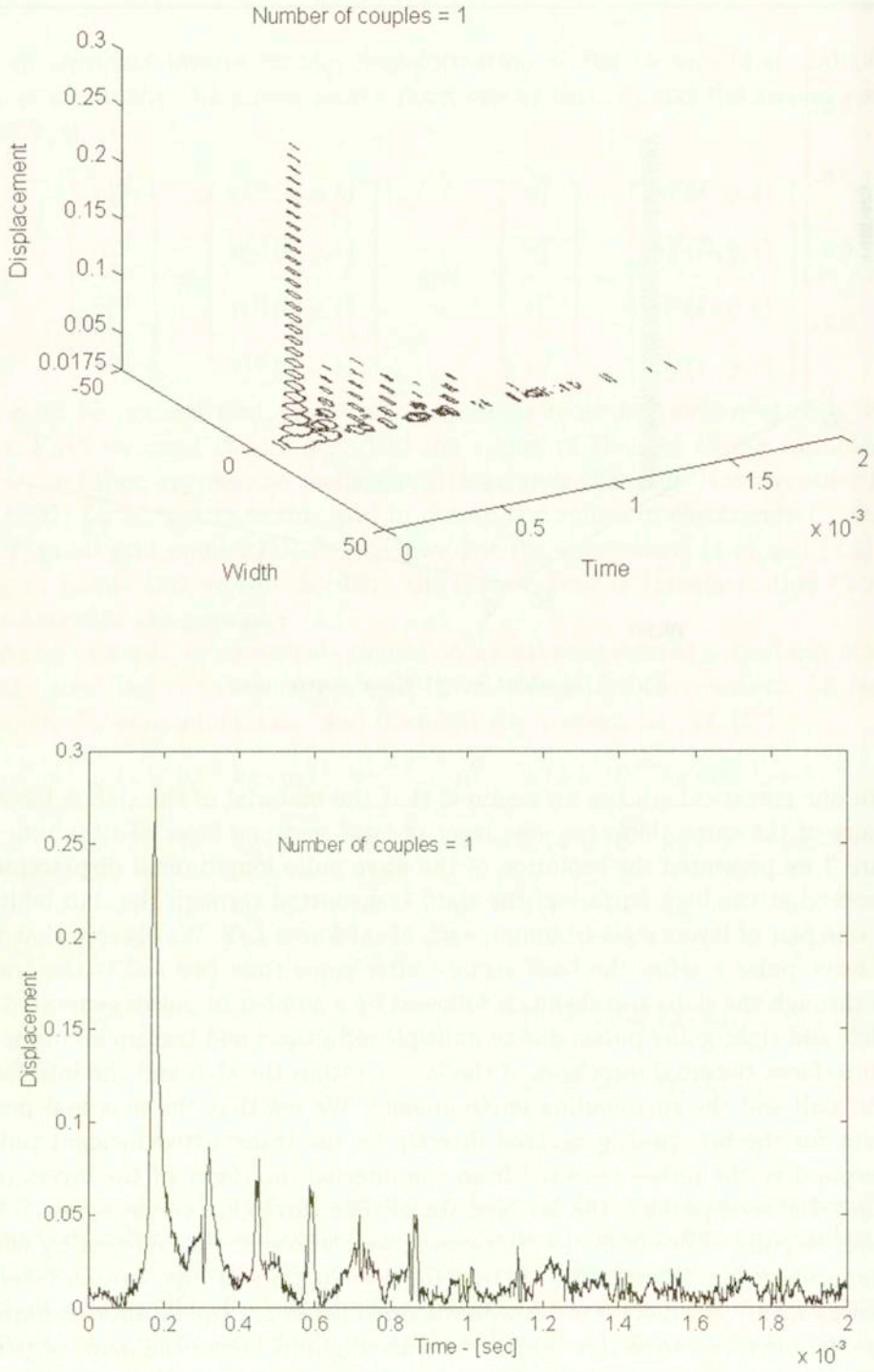

FIG. 3. Transmitted longitudinal wave pulse (n=1 pair of layers). Two-dimensional picture One-dimensional cross-section at the middle of pulse.

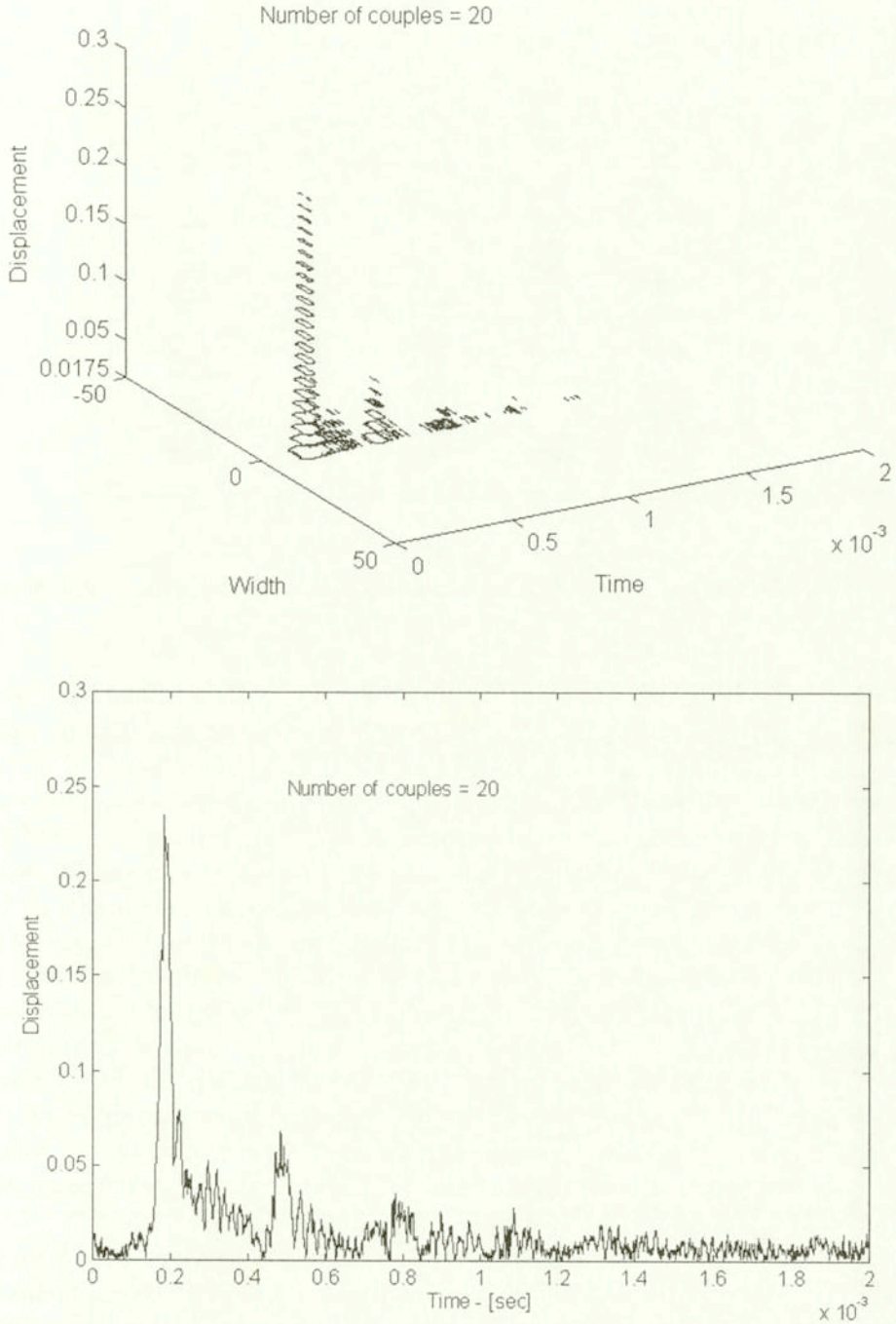

FIG. 4. Transmitted longitudinal wave pulse (n=20 pairs of layers). a) Two-dimensional picture; b) One-dimensional cross-section at the middle of pulse.

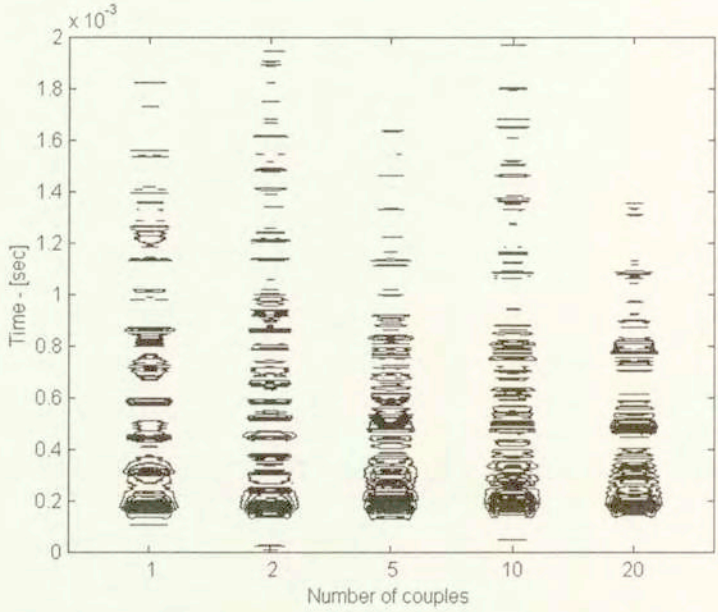

FIG. 5. Map of the transmitted longitudinal wave pulses for $n = 1, 2, 5, 10$ and 20 pairs of layers.

Our calculations and, what follows, the accuracy of the obtained picture is strongly restricted by the number the grid points possible for calculations within a realistic time (this is the time restriction for the applied algorithm of two-dimensional Fast Fourier Transformation – [8]). For this reason we cannot study the transmission of pulses for a greater number of layers (too few points of grid at every layer). However, calculations for the one-dimensional model [4], where only one-dimensional Fourier transformations are needed, show two effects. One is the convergence of the shape of the transmitted wave pulse to the shape of the incident pulse if the number of layers inside the slab grows (in our example the transmitted pulse is of the constant amplitude). The localisation of the wave pulse is another effect generated by a multi-layered medium. We can see that the wave pulse goes through the homogenised slab (the slab with many pairs of very narrow layers) longer than the sum of travel times through the material components. On the other hand, we can observe the concentration of displacements (and, what it follows, stresses) due to summation of several pulses, reflected and transmitted inside the slab, what can be called the localisation of stresses.

As it is known, the incident longitudinal wave pulse generates, except for the reflected and transmitted longitudinal wave pulses, also the transversal (reflected and transmitted) pulses. Figures 6a, 6b show the transmitted transversal pulses measured at the back surface of the layered slab for $n = 1$ and $n = 20$ pairs of layers, respectively, and for two instants of time. It is seen that the transverse

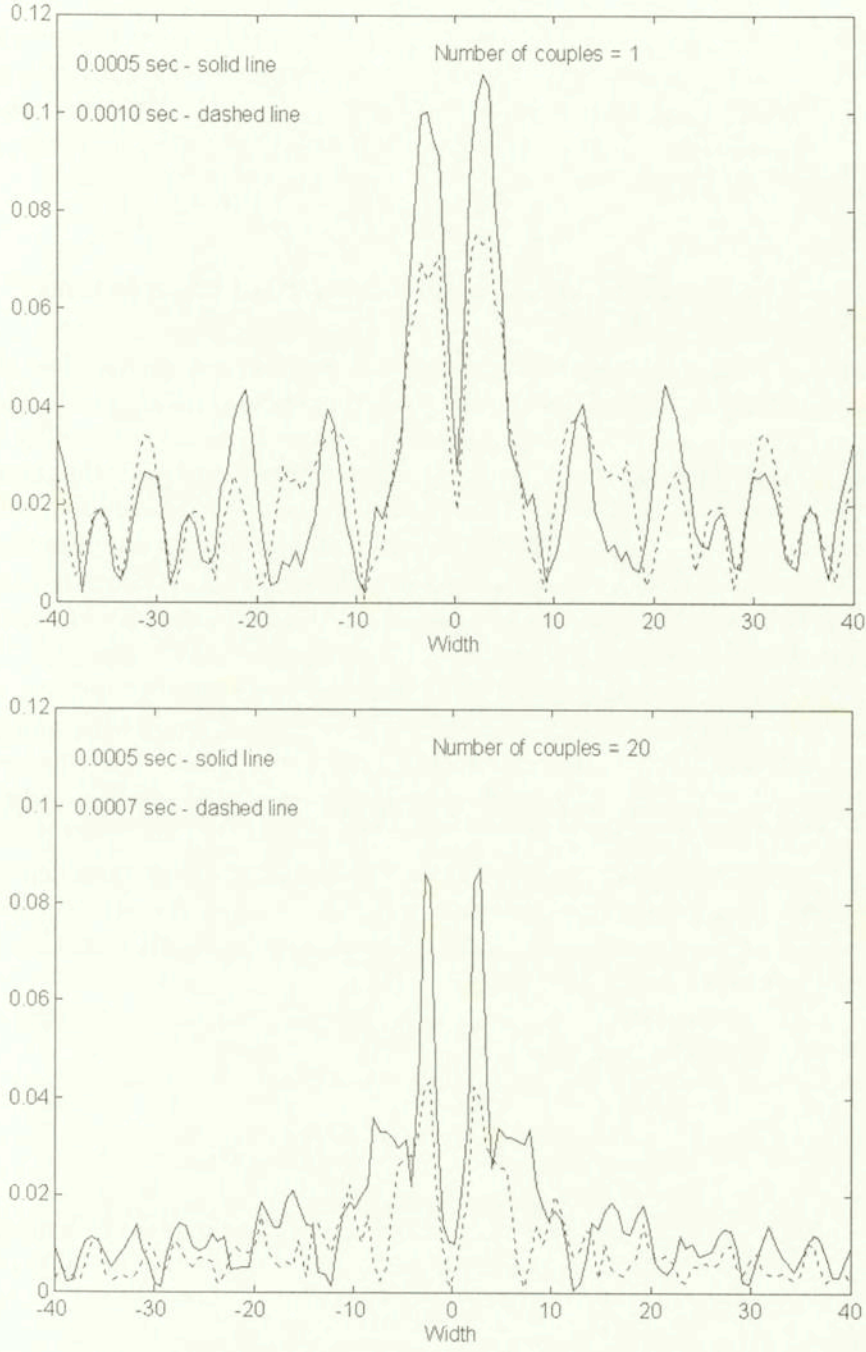

FIG. 6. Transmitted transversal wave pulse. a) $n = 1$ pair of layers. The cross-sections at time $t = 0.0005$ sec (solid line) and $t = 0.001$ sec (dashed line); b) $n = 20$ pairs of layers. The cross-sections at time $t = 0.0003$ sec (solid line) and $t = 0.0007$ sec (dashed line).

displacements are generated on the edges of the longitudinal wave. Unfortunately, in our calculations (to rare grid) we obtain a very high level of numerical noise, so too it is hard to conclude about evolution of this wave in time. We can only say that transverse pulses are proportional to the longitudinal ones, so they have peaks at the same time.

5. Wave pulses effect on fatigue fracture at interface imperfections

Two-dimensional analysis of the wave pulse propagation in stratified elastic medium provides the normal and shear stress components within a slab. Though the layers are assumed to be perfectly bonded, there is still a probability of an inclusion to be present at the interface. Assuming the inclusion in the form of a crack, the questions arise what a single impulse can be applied to the slab without initiating the fracture of the bond or how many and how intense impulses the slab can sustain without failing due to the fatigue damage. Both problems require the fracture mechanics theory for an interface in an idealised infinite plane between two linear elastic materials to be used.

For two perfectly bonded materials of different mechanical properties, shear modules, G_j, and Poisson's ratios, ν_j, $j = 1, 2$, say, both the tensile and shear stresses exist always at the crack tip affecting the fracture mode. There are several papers devoted to the interface crack growth initiation in dissimilar media, e.g. [9, 10, 13] and many others. It is commonly recognised that the crack branches initially from an interface in the softer of the two materials, at an angle depending on the stress components and material properties. As the branch extends, the crack tends to return to a path parallel to the interface with a driving force similar to that of an unbranched crack, cf. [6].

In order to prevent the propagation of an interface crack, the following criterion, cf. [13], can be adopted:

$$K_{\theta_{\max}} = B(\theta_0, \varepsilon, \gamma) \cdot \frac{\sqrt{K_1^2 + K_2^2}}{2 \cdot \cosh(\pi\varepsilon)} < K_{\mathrm{IC}} \tag{5.1}$$

where K_1 and K_2 are the real and imaginary parts of the complex stress intensity factor, $K = K_1 + iK_2$, for an interface crack, and ε is the bimaterial constant defined by

$$\varepsilon = \frac{1}{2\pi} \cdot \ln\left(\frac{\kappa_1/G_1 + 1/G_2}{\kappa_2/G_2 + 1/G_1}\right) \tag{5.2}$$

with $\kappa_j = 3 - 4\nu_j$ for plane strain or $\kappa_j j = (3 - \nu_j)/(1 + \nu_j)$ for plain stress problem, $j = 1, 2$. The parameter $\gamma = \tan^{-1}(K_2/K_1)$ and K_{IC} denotes the fracture

toughness. The criterion (5.1) is derived for $|\varepsilon| < 0.1$ under an assumption that the direction for which the circumferencial stress component, σ_θ, of the stress field in the polar co-ordinate system originated in the crack tip reaches its maximum, coincides with the initial crack growth direction, θ_0, i.e.

$$\frac{\partial \sigma_\theta}{\partial \theta} = 0. \tag{5.3}$$

The branching direction, θ_0, can be eventually calculated from the following equation:

$$\begin{aligned} \varepsilon \cdot e^{\varepsilon(\theta-\pi)} \cdot \left[2\cos\left(\frac{\theta}{2}+\gamma\right) - (\cos\theta + 2\varepsilon\sin\theta)\cdot\cos\left(\frac{\theta}{2}-\gamma\right)\right] \\ - e^{\varepsilon(\theta-\pi)} \cdot \left[\sin\left(\frac{\theta}{2}+\gamma\right) - (\sin\theta - 2\varepsilon\cos\theta)\cdot\cos\left(\frac{\theta}{2}-\gamma\right)\right. \\ \left. - \frac{1}{2}(\cos\theta - 2\varepsilon\sin\theta)\cdot\sin\left(\frac{\theta}{2}-\gamma\right)\right] \\ - e^{-\varepsilon(\theta-\pi)} \cdot \left[\varepsilon\cos\left(\frac{3\theta}{2}+\gamma\right) + \frac{3}{2}\sin\left(\frac{3\theta}{2}+\gamma\right)\right] = 0. \end{aligned} \tag{5.4}$$

The function term $B(\theta, \varepsilon, \gamma)$ occurring in (5.1) results from the formula for σ_θ and has got the following form:

$$\begin{aligned} B(\theta,\varepsilon,\gamma) = e^{\varepsilon(\theta-\pi)} \cdot \left[2\cos\left(\frac{\theta}{2}+\gamma\right) - (\cos\theta + 2\varepsilon\sin\theta)\cdot\cos\left(\frac{\theta}{2}-\gamma\right)\right] \\ + e^{-\varepsilon(\theta-\pi)} \cdot \cos\left(\frac{3\theta}{2}+\gamma\right). \end{aligned} \tag{5.5}$$

The components, K_1 and K_2, of the complex stress intensity factor are determined numerically, cf. [13], or analytically using the following formulae based on those derived in [9], say,

$$\begin{aligned} K_1 = \sigma\,[\cos(\varepsilon\log 2a) + 2\varepsilon\sin(\varepsilon\log 2a)] \\ + \tau\,[\sin(\varepsilon\log 2a) - 2\varepsilon\cos(\varepsilon\log 2a)]\cdot\sqrt{\pi\cdot a}, \end{aligned} \tag{5.5}_1$$

$$\begin{aligned} K_2 = \tau\,[\cos(\varepsilon\log 2a) + 2\varepsilon\sin(\varepsilon\log 2a)] \\ - \sigma\,[\sin(\varepsilon\log 2a) - 2\varepsilon\cos(\varepsilon\log 2a)]\cdot\sqrt{\pi\cdot a}. \end{aligned} \tag{5.5}_2$$

Substituting the latter into (5.1), the fracture criterion takes the simple form as follows

$$K_{\theta_{\max}} = B(\theta_0, \varepsilon, \gamma) \cdot \frac{\sqrt{(\sigma^2 + \tau^2) \cdot (1 + 4\varepsilon^2)}}{2 \cdot \cosh(\pi\varepsilon)} \cdot \sqrt{\pi \cdot a} < K_{\mathrm{IC}}. \tag{5.6}$$

In analogy to the fatigue crack growth modelling in homogenous materials the amplitude of the equivalent one-dimensional stress intensity factor, $\Delta K_{\theta_{\max}}$, cf. (5.6), related to the amplitudes of the stress components, $\Delta\sigma$ and $\Delta\tau$, can be likely used in a fatigue crack evolution equation in the form $da/dn = f(\Delta K_{\theta_{\max}})$ or more specifically, recalling the Paris-Erdogan equation, $da/dn = C \cdot \Delta K^m_{\theta_{\max}}$ with C and m as some constants. The further the crack penetrates into the material, the weaker becomes the effect of the interface on the crack propagation, in particular on its direction. It seems to be rational to assume the fatigue plastic zone size, r_p, as a characteristic distance between the crack tip and the interface when $\varepsilon = 0$ should be admitted in calculation of a new crack path direction, θ_0, cf. (5.4), and of the stress intensity factor amplitude, $\Delta_{\theta_{\max}}(\theta_0, 0, \gamma)$, cf. (5.6). Since the problem of plastic zone size calculation for the interface crack is rather obscure, cf. [10], the approximate equation

$$r_p \approx \frac{\pi}{2} \cdot (1 + 4\varepsilon^2) \cdot \frac{\Delta\sigma^2 + \Delta\tau^2}{\sigma_0^2} \cdot a \tag{5.7}$$

where σ_0 denotes the yield stress, can be adopted to indicate the moment when the interface effect might be neglected.

6. Conclusions

In the paper we considered the model of the stratified medium – the slab built of some number of isotropic, homogeneous elastic layers. Such a medium, globally, is both anisotropic and nonhomogeneous. It is known that if we assume the global thickness of the slab of layers to be constant but increase infinitely the number of layers inside the slab, we perform a homogenisation procedure so that after this process, the slab becomes homogeneous but remains anisotropic (locally and globally transversally isotopic). The homogenisation effect, observed during the numerical experiment, can be confirmed analytically, both for periodic and random layered media [5]. To prove this fact we can calculate the limit transition matrix (using the law of large numbers for products of matrices in the random case, cf. [1]). The elastic properties of the obtained effective medium are described by a tensor whose 5 elements are independent [3, 5]. However, as we have seen from the considerations of Section 2, for the description of the elastic waves in the case of the plane state of displacement we need only four elastic constants. Two constants are needed in the anti-plane state of displacement (one of them being different than that in the plane state). This statement remains valid both

in the dynamic nonstationary case, studied in this paper, and in the stationary one [3].

In this paper we have presented an analytical method of solution of the dynamic wave problem in a layered medium. However, to express the resulting waves (reflected and transmitted) generated by some incident pulse in an explicit way some numerical calculations are necessary. The two-dimensional inverse Fourier transform using the Fast Fourier Transform algorithm [8] appears to be the most effective numerical tool to perform the calculations.

Because of the multiple reflections and possible localisation of the stress pulses in the stratified media, the fatigue fracture effects must not be neglected in reliability assessment of structures made of such composites. Therefore some remarks on the interface crack propagation were given to point out the problem and suggest a possible approach to deal with it. Some numerical calculation and, what is of the greatest importance, experimental verification are now going on and will be reported elsewhere in the future.

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