On rate and gradient-dependence of solids as dynamical systems

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THIS PAPER AIMS at presenting a mathematical background of material instability problems such like strain localization or flutter, as an application of the theory of dynamical systems. The basic field equations of the solid continuum are the kinematic equations, the Cauchy equations of motion and the constitutive equations. This system of equations is completed with initial and boundary value conditions and can define a dynamical system. Then, a condition of material stability can be obtained using Lapunov's indirect method. Also the basic material instability modes can be classified as static (divergence type) or dynamic (Hopf) bifurcations of dynamical systems. Such formulation gives a mathematical interpretation of rate and gradient-dependence in the constitutive equations and mesh dependence in numerical studies, pointing out their close relationship to the structure of the critical eigenspace of an operator defined by the dynamical system.

1. Introduction

IN THE RECENT YEARS several new results of the theory of dynamical systems [1, 18] have already been successfully used in various fields of mechanics [5, 16]. The main aim of this paper is to analyze the effect of rate and gradient-dependence in material instability and postlocalization by considering solid continua as dynamical systems [4, 6]. This kind of investigation is closely related to the perturbation analysis [9, 19] as it will be explained in Section 3.

In the theory of dynamical systems, the linear concept of the loss of stability of a state of the system means that the real part of certain eigenvalues of the linear operator describing its behavior changes sign. The eigenvectors connected with them are used in applications to the critical eigenmodes [16]. In a nonlinear case, the postbifurcation can be studied and described analytically using these critical eigenmodes.

Unfortunately, for the classical setting [13] there is no possibility to obtain specific critical eigenmodes at the onset of material instability. On the other

hand, in the finite element calculation of material instability problems, the classical formulation of the basic equations of solid continua results in a definite mesh dependence [7, 14, 15]. These are very similar phenomena. In those papers the mesh dependence was eliminated by the inclusion of rate-dependence or nonlocality (gradient effects) into the constitutive equations.

In this paper we study how and when the inclusion of rate and gradient dependent terms into the constitutive equations changes the structure of critical eigenmodes, and we show how the postlocalization can be studied for a nonlinear constitutive equation.

The second section presents the basic equations for the solid body. We neglect geometrical nonlinearities, but include the nonlinearity in the constitutive equation. The equations will be transformed into the velocity field, because such form is convenient for our investigation. The third section starts with a survey of the basic notions of the theory of dynamical systems and formulates a linear stability condition for a state of the body based on the Lapunov stability definition. These conditions are formulated as linear boundary value problems. By solving them, the eigenfunctions can be considered as critical eigenmodes. In case of a unique critical eigenmode, a static bifurcation investigation is performed for the nonlinear equation.

In section four the effect of rate and gradient-dependence is studied in uniaxial cases. These are mostly linear problems, but at the end we show how the critical eigenmode can be used for the analysis of a static post-localization in presence of a material nonlinearity.

2. Basic equations

In case of small strains the kinematic equation [11] is

$$(2.1) \qquad \epsilon = \frac{1}{2} \left(\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u} \right),$$

where ϵ is the strain tensor, $\mathbf{u}$ is the displacement vector and $\circ$ denotes a diadic product. The equation of motion [11] without body forces is

$$(2.2) \qquad \rho \ddot{\mathbf{u}} = \sigma \nabla,$$

where ρ is the mass density and σ denotes the symmetric stress tensor.

Let the constitutive equation have the form

$$(2.3) \qquad F\left(\dot{\sigma}, \epsilon, \dot{\epsilon}, \nabla^2 \epsilon, \nabla \epsilon, \sigma \right) = 0$$

representing both the rate-dependent and gradient effects. Equation (2.3) is a combination of the rate-dependent second gradient theory [17] and of the first

gradient constitutive equation suggested in [2], which is developed on the basis of the wave dynamics [3] and of several experimental results [2].

By studying the stability of a state S^0 described by $\sigma^0, \epsilon^0 \dots$ the constitutive equation can be linearized at S^0. Let the new variables $\bar{\sigma} = \sigma - \sigma^0$, $\bar{\epsilon} = \epsilon - \epsilon^0 \dots$ be introduced for arbitrarily small perturbations. Because at state S^0 the variables satisfy (2.3), we may assume that the linearized constitutive equation is

$$(2.4) \qquad \dot{\bar{\sigma}} = \mathbf{C}^1 \bar{\epsilon} + \mathbf{C}^2 \dot{\bar{\epsilon}} + \mathbf{C}^3 \nabla^2 \bar{\epsilon} + \mathbf{C}^4 \nabla \bar{\epsilon} + \mathbf{C}^5 \bar{\sigma}.$$

Now equations (2.1), (2.2) and (2.4) form the basic equations for the stability investigation of state S^0. These equations should be transformed into the velocity field **v**. For the sake of simplicity the bars are omitted in the following derivations but all equations concern the small perturbations of state S^0, thus all the calculations are performed in a sufficiently small neighbourhood of S^0.

From (2.2) and by using the rate form of (2.1) and (2.4), the basic equation is

$$(2.5) \qquad 2\rho \, \ddot{\mathbf{v}} = \mathbf{C}^1 \left(\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v} \right) \nabla + \mathbf{C}^2 \left(\dot{\mathbf{v}} \circ \nabla + \nabla \circ \dot{\mathbf{v}} \right) \nabla$$

$$+ \mathbf{C}^3 \nabla^2 \left(\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v} \right) \nabla + \mathbf{C}^4 \nabla \left(\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v} \right) \nabla + \mathbf{C}^5 \rho \ddot{\mathbf{v}}.$$

The initial and boundary conditions assumed by the mechanical problem under consideration are also needed. In the following considerations, the stability investigation of state S^0 will be based on (2.5).

3. Dynamical systems and bifurcations

In operator form Eq. (2.5) reads

$$(3.1) \qquad \ddot{v} = F^1 v + F^2 \dot{v} + F^3 \ddot{v}.$$

Here $v = (v_1, v_2, v_3)$ is a vector of the coordinates of the velocity field satisfying the boundary conditions and F^1, F^2 and F^3 are linear differential operators defined by the right-hand side of (2.5). Equation (3.1) can be considered as an infinite-dimensional dynamical system.

The stability of state S^0 of the continuum is defined by the Lapunov stability of a solution $v^0(t)$ of (3.1). That is, a state represented by $v^0(t)$ is stable, when the perturbed velocity field $v^0(t) + \bar{v}(t)$ remains sufficiently close to the unperturbed one. Such definitions are also used in solid mechanics [11, 12]. The stability investigation of $v^0(t)$ starts with a transformation into a local form at that solution by substituting

$$v(t) = v^0(t) + \bar{v}(t)$$

into (3.1):

$$(3.2) \qquad \ddot{v}^0 + \ddot{\bar{v}} = F^1 \left(v^0 + \bar{v} \right) + F^2 \left(\dot{v}^0 + \dot{\bar{v}} \right) + F^3 \left(\ddot{v}^0 + \ddot{\bar{v}} \right).$$

While v^0 is a solution of (3.1) and F^1, F^2 and F^3 are linear operators, the first terms of all parts in (3.2) are equal, thus the equation of motion (3.2) for the perturbation $\bar{v}(t)$ has the same form as (3.1). Then (3.2) can be transformed into a system of first order equations by introducing new variables and vectors

$$y_1 = \bar{v}_1, \quad \ldots, \quad y_3 = \bar{v}_3, \quad y_4 = \dot{\bar{v}}_1, \quad \ldots, \quad y_6 = \dot{\bar{v}}_3, \quad y_7 = \ddot{\bar{v}}_1, \quad \ldots, \quad y_9 = \ddot{\bar{v}}_3$$

$$y_\alpha, \quad (\alpha = 1,\ldots,3), \quad y_\beta, \quad (\beta = 4,\ldots,6), \quad y_\psi, \quad (\psi = 7,\ldots,9).$$

The transformed equations are

(3.3) $$\dot{y}_\alpha = y_\beta,$$

(3.4) $$\dot{y}_\beta = y_\psi,$$

(3.5) $$\dot{y}_\psi = F^1 y_\alpha + F^2 y_\beta + F^3 y_\psi.$$

The stability properties are determined by the eigenvalues of linear operator $\hat{F}$ defined by the right-hand sides of (3.3), (3.4) and (3.5),

$$\hat{F}(y_\alpha, y_\beta, y_\psi) = (y_\beta, y_\psi, F^1 y_\alpha + F^2 y_\beta + F^3 y_\psi).$$

Using Lapunov's indirect method [8] v^0 is asymptotically stable, when the real parts of all eigenvalues of $\hat{F}$ are negative. In case of zero real parts, the system is on the stability boundary. The characteristic equation of $\hat{F}$ reads

(3.6)
$$\begin{aligned} \lambda y_\alpha &= y_\beta, \\ \lambda y_\beta &= y_\psi, \\ \lambda y_\psi &= F^1 y_\alpha + F^2 y_\beta + F^3 y_\psi. \end{aligned}$$

By substituting the first two equations of (3.6) into the third one, equation

(3.7) $$\lambda^3 y_\alpha - \lambda^2 F^3 y_\alpha - \lambda F^2 y_\alpha - F^1 y_\alpha = 0$$

is obtained. The condition of asymptotic stability is $\mathrm{Re}\lambda_i < 0$, $i = 1\ldots$ for all λ_i satisfying (3.7).

The typical ways of loosing stability are the following cases: when (SB) a real λ_c or (DB) the real part of a pair of complex conjugate λ_{c1} and $\lambda_{c2}(= \bar{\lambda}_{c1})$ changes sign, while all the others satisfy $\mathrm{Re}\lambda_i < 0$, $i \neq c$ and $i \neq c1, c2$, respectively. Thus the loss of stability can either be a generic static (SB) or dynamic (DB) bifurcation [6]. In case of a (SB), Eq. (3.7) has a zero eigenvalue $\lambda_c = 0$. From (3.7) the condition of (SB) is

(3.8) $$F^1 y_\alpha = 0.$$

This phenomenon is also called the divergence instability or the onset of strain localization [13], because also the uniqueness of the solution v^0 is lost and other, localized nontrivial solutions appear.

At (DB) the eigenvalues are imaginary values, thus the necessary condition is

$$(3.9) \qquad \left(\left(F^3\right)^{-1} F^1 + F^2\right) y_\alpha = 0.$$

The main difference between instabilities (SB) and (DB) is that at (DB) the uniqueness of v^0 remains valid, but Lapunov stability is lost.

While (2.5) and (3.1) show that F^1 depends on the so-called hardening parameter and F^2 on the rate-sensitivity, this classification is similar to that in [19], where (SB) is called the strain-hardening type and (DB) is the rate sensitivity type.

Notice that all functions y_α should also satisfy the boundary conditions, thus expressions (3.8), (3.9) are partial differential equations with those boundary conditions. To obtain the critical eigenfunctions (eigenmodes), these linear boundary value problems should be solved. Unfortunately, in a general case this step cannot be done analytically. Instead of solving it we could use two kinds of simplifications.

Firstly, we could restrict ourselves to one-dimensional problems. Then an analytical solution can easily be calculated (see the example in the following).

Secondly, in the three-axial case special solutions could be studied using the method called the perturbation technique. It is widely used (see for instance [9, 19]) to omit the boundary value problem. Then the treatment is restricted to the study of the functions

$$q_k(t) \exp(in_p^k x_p),$$

substituted into the equation of motion. Then conditions (3.8), (3.9) turn out to be systems of algebraic equations. A detailed study of the linear case is given in [6].

At the end of this section the nonlinear post-localization will be studied in case of an (SB) of a stationary (or steady state) solution. A solution of (3.3) and (3.5) is called stationary, when

$$\dot{y}_\alpha = \dot{y}_\beta = \dot{y}_\psi = 0.$$

Let us study what happens with such solutions at a static bifurcation. In the investigation also the nonlinear terms $\hat{N}(y_\alpha, y_\psi)$ are necessary. When also nonlinear terms are added to Eqs. (3.3) and (3.5), for the stationary solutions

$$(3.10) \qquad 0 = F^1 y_\alpha + N(y_\alpha)$$

is obtained, where $N(y_\alpha) = \hat{N}(y_\alpha, 0)$. Assume that F^1 depends on a (for example loading) parameter μ, and at $\mu = 0$ condition (3.8) of the static bifurcation is satisfied by the eigenmode y_α^0,

$$(3.11) \qquad F^1\Big|_{\mu=0} y_\alpha^0 = 0.$$

Defining $\bar{F}^1(\mu) = F^1 - F^1|_{\mu=0}$, Eq. (3.10) assumes the form

(3.12)
$$0 = \left(\bar{F}^1(\mu) + F^1|_{\mu=0}\right) y_\alpha + N(y_\alpha).$$

In static bifurcation theory [5] in a small neighbourhood of v^0 (or state S^0), the nontrivial solution can be searched for in the form

$$y_\alpha = q y_\alpha^0,$$

where q is a small real number. By substituting this form into (3.12), relation

(3.13)
$$0 = q\bar{F}^1(\mu)y_\alpha^0 + N\left(q y_\alpha^0\right)$$

is obtained because of (3.11). Introducing a scalar product $< .,. >$, from (3.13) an approximation of the bifurcation equation [16]

(3.14)
$$0 = q\left\langle y_\alpha^0, \bar{F}^1(\mu)y_\alpha^0\right\rangle + \left\langle y_\alpha^0, N\left(q y_\alpha^0\right)\right\rangle$$

is obtained, which is a nonlinear algebraic equation for q. By performing power series expansions and considering only the first few terms, it can be solved for $q = q(\mu)$. Then for a sufficiently small μ, the nontrivial solution is

$$y_\alpha = q(\mu)y_\alpha^0.$$

4. Rate and gradient dependence

In the following parts the static and dynamic bifurcations of solid bodies will be studied in a one-dimensional example. We will also analyse the static post-bifurcation (post-localization) of a second gradient dependent nonlinear material. Let a rod of length L be considered. Several kinds of rate-dependence of the constitutive equations will also be investigated. These are second gradient-dependent and gradient-independent materials. We also study two types of instability problems of rate and first gradient dependent materials.

4.1. A linear second gradient-dependent material

In case of a second gradient-dependent material, the constitutive equation in rate form [17]

(4.1)
$$\dot{\sigma} = \bar{c}_1\dot{\epsilon} + \bar{c}_2\ddot{\epsilon} - \bar{c}_3\frac{\partial^2\dot{\epsilon}}{\partial x^2}$$

is a widely used one, based on the second gradient theory and succesfully used in static post-localization. Equations (3.4) and (3.5) in this case are

(4.2)
$$\dot{y}_1 = y_2,$$

$$(4.3) \qquad \dot{y}_2 = \left(c_1 \frac{\partial^2}{\partial x^2} - c_3 \frac{\partial^4}{\partial x^4} \right) y_1 + c_2 \frac{\partial^2}{\partial x^2} y_2,$$

where $c_i = \dfrac{\bar{c}_i}{\rho}$, $i = 1, 2, 3$. Now, the characteristic Eq. (3.7) is a second order one

$$(4.4) \qquad \lambda^2 y_1 - \lambda c_2 \frac{\partial^2}{\partial x^2} y_1 - \left(c_1 \frac{\partial^2}{\partial x^2} - c_3 \frac{\partial^4}{\partial x^4} \right) y_1 = 0.$$

In case of homogeneous boundary conditions, the eigenfunctions are

$$(4.5) \qquad y_1 = e^{i\alpha_k x}, \quad \text{where} \quad \alpha_k = \frac{k\pi}{L} \quad (k = 1, ...)$$

and the eigenvalues are

$$(4.6) \qquad \lambda_{1,2,k} = \frac{-c_2 \alpha_k^2 \pm \sqrt{c_2^2 \alpha_k^4 - 4\alpha_k^2 \left(c_3 \alpha_k^2 + c_1 \right)}}{2}.$$

When c_1 and c_2 are positive, the real parts of all the eigenvalues are negative. Thus the state is stable, if $c_2 > 0$ and

$$(4.7) \qquad \left(c_3 \alpha_k^2 + c_1 \right) > 0.$$

The (SB) loss of stability happens, when

$$(4.8) \qquad \left(c_3 \alpha_k^2 + c_1 \right) = 0.$$

Then

$$(4.9) \qquad \alpha_k = \alpha_{k*} = \sqrt{-\frac{c_1}{c_3}}.$$

In this case the only function of form (4.5) satisfying (4.9) is

$$(4.10) \qquad v = e^{i\sqrt{-\frac{c_1}{c_3}} x}.$$

Obviously, when for some k

$$(4.11) \qquad \alpha_k < \alpha_{k*},$$

one of the eigenvalues $\lambda_{1,2,k}$ has a positive real part. For that k this implies instability.

Since c_1 is the tangent of the stress-strain diagram at state S^0 and during a quasistatic loading process it gets increasingly negative values on the softening side, the first critical α_k is at $k = 1$. In case of the so-called adiabatic localization [13] L tends to infinity. Then the instability condition (4.8) can be satisfied by

any arbitrary small real α, thus the state loses stability, when c_1 assumes negative values. Thus the stability boundary for the adiabatic case is $c_1 = 0$.

When the loss of stability is of type (DB), the condition is

$$c_2 = 0.$$

Before the loss of stability $\mathrm{Re}(\lambda_{1,2,k}) < 0$, thus (4.7) should be satisfied. Assume that it remains true during the dynamic loss of stability. Then at $c_2 = 0$ all eigenvalues are imaginary numbers

$$(4.12) \qquad \lambda_{1,2,k} = \pm i\alpha_k \sqrt{(c_3\alpha_k^2 + c_1)}$$

and each of them can be attached to a critical eigenmode. Unfortunately, at $\alpha_k = \alpha_{k*}$ expression (4.12) yields a zero eigenvalue, thus a coexistent (SB) and (DB) instability happens.

From (4.6) we can see that rate-independence for such a material is equivalent with the (DB) condition. Thus such material is always on the stability boundary (of course it may be Lyapunov stable but not asymptotically stable [18]). The "real" loss of stability happens as an additional (SB), if $c_1 = 0$. Then we obtain again coexistent (SB) and (DB) instabilities. (For details see [6].)

4.2. The gradient-independent case

Expression (4.6) shows the differences between the gradient-dependent and independent cases. When the material is gradient-independent, $c_3 = 0$. Then from (4.6)

$$(4.13) \qquad \lambda_{1,2,k} = \frac{-c_2\alpha_k^2 \pm \sqrt{c_2^2\alpha_k^4 - 4\alpha_k^2 c_1}}{2}.$$

The condition of (SB) is $c_1 = 0$, because then (4.13) implies

$$(4.14) \qquad \lambda_{1,2,k} = \frac{-c_2\alpha_k^2 \pm |c_2|\alpha_k^2}{2},$$

that is,

$$(4.15) \qquad \lambda_{1,k} = \frac{-c_2 + |c_2|}{2}\alpha_k^2, \quad k = 1, 2, \ldots$$

$$(4.16) \qquad \lambda_{2,k} = \frac{-c_2 - |c_2|}{2}\alpha_k^2, \quad k = 1, 2, \ldots$$

By comparing (4.8) and (4.14) we see, that the main difference is that (4.8) defines a critical $k = k*$ (see (4.9)) and consequently, a critical eigenmode $e^{i\alpha_{k*}x}$ for the perturbation. In (4.14) all values k, $k = 1, 2 \ldots$ and all perturbations

$e^{i\alpha_k x}$ are critical when $c_1 = 0$. In other words, for gradient-independent constitutive equation all wavelengths are critical. At the postbifurcation investigation, the nontrivial solutions were searched for as linear combination of the critical eigenmodes. Such study cannot be performed for rate-independent constitutive equation because of the infinite number of critical eigenmodes. Moreover, while $c_2 > 0$ suffices to ensure stability for eigenvalues (4.16), the other group (4.13) results in a zero eigenvalue with infinite multiplicity.

In case of (DB) instability, the necessary condition is $c_2 = 0$ and for the eigenvalues

$$\lambda_{1,2,k} = \pm i\sqrt{c_1}\alpha_k.$$

Contrary to the previous case, now (SB) and (DB) instabilities are distinct phenomena.

When rate-dependence is omitted we again have the permanent (DB) as in part 4.1.

4.3. The effect of material nonlinearity

In this subsection a nonlinear constitutive equation proposed in [20] is used. This one contains both second gradient-dependent and rate-dependent terms

$$(4.17) \qquad \dot{\sigma} = c_1\dot{\epsilon} + c_2\ddot{\epsilon} - c_3\frac{\partial^2\dot{\epsilon}}{\partial x^2} + c_4\left(\frac{\partial\dot{\epsilon}}{\partial x}\right)^2$$

for the adiabatic postlocalization investigation in the one-dimensional case. Assume that the loss of stability of state S^0 happens at c_{10}. A small bifurcation parameter $0 < \mu \ll 1$ is introduced,

$$(4.18) \qquad c_1 = c_{10} - \mu.$$

Using (4.17), (4.18) and the one-dimensional form of (2.1) and (2.2), the equation of motion for the velocity field is

$$(4.19) \qquad \rho\ddot{v} = \left(c_{10}\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)v - \mu\frac{\partial^2}{\partial x^2}v + c_2\frac{\partial^2}{\partial x^2}\dot{v} + c_4\left(\frac{\partial^3 v}{\partial x^3}\right)^2.$$

While the localization is a static bifurcation [6], the postbifurcation investigation can be restricted to the steady state solutions $\ddot{v} = \dot{v} = 0$ of (4.19). Then instead of (4.19), equation

$$(4.20) \qquad \left(c_{10}\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4}\right)v - \mu\frac{\partial^2}{\partial x^2}v + c_4\left(\frac{\partial^3 v}{\partial x^3}\right)^2 = 0$$

is used.

In the linear study of the previous subsection at the loss of stability, the velocity function in form (4.10) was obtained. Thus, as the first critical eigenmode, function $\sin(\alpha x)$ can be identified. Similarly to the general treatment, the bifurcated nontrivial solution of (4.20) can be searched for as a linear combination of the critical eigenmodes. Now there is a unique critical eigenmode, thus

$$v^c = q\sin(\alpha x),$$

where $|q| \ll 1$ and α satisfies (4.9). Function v^c can be substituted into (4.20) and then the scalar product (the right-hand side of (3.14)

$$g(q,\mu) = \int\limits_0^L \left(\left(\left(c_{10}\frac{\partial^2}{\partial x^2} - c_3\frac{\partial^4}{\partial x^4} - \mu\frac{\partial^2}{\partial x^2} \right) q\sin(\alpha x) \right) \sin(\alpha x) \right.$$
$$\left. + c_4\left(\frac{\partial^3 q\sin(\alpha x)}{\partial x^3} \right)^2 \right) \sin(\alpha x)dx$$

defines function $g(q,\mu)$ for the approximate bifurcation equation

(4.21) $$g(q,\mu) = 0.$$

Solving (4.21), we obtain

(4.22) $$q = -\frac{3\mu}{4\pi c_4\alpha^4}$$

and thus a transcritical bifurcation is obtained and the nontrivial solution is

$$v^c = -\frac{3}{4\pi c_4\alpha^4}\mu\sin(\alpha x),$$

or, using (4.9),

(4.23) $$v^c = -\frac{3c_3^2}{4\pi c_4 c_{10}^2}\mu\sin\left(x\sqrt{-\frac{c_{10}}{c_3}} \right).$$

4.4. First gradient effects

Let us study now the first gradient effects. Then the constitutive equation has the generalized form

(4.24) $$\dot{\sigma} = \bar{d}_1\dot{\epsilon} + \bar{d}_2\frac{\partial\epsilon}{\partial x} + \bar{d}_3\epsilon + d_4\sigma,$$

which was studied in connection with acceleration waves, both theoretically and experimentally in [3], and contains a first gradient term.

Firstly, let us set $d_4 = 0$, which is an experimental result for copper [2]. Then the characteristic equation (3.7) is

$$(4.25) \qquad \lambda^3 y_1 - \lambda d_1 \frac{\partial^2}{\partial x^2} y_1 - d_3 \frac{\partial^2}{\partial x^2} y_1 - d_2 \frac{\partial^3}{\partial x^3} y_1 = 0,$$

where $d_i = \dfrac{\bar{d}_i}{\rho}$, $i = 1, 2, 3$. For homogeneous boundary conditions, as before Eq. (4.5) can be substituted into (4.25) and then a complex equation

$$(4.26) \qquad \lambda^3 + \lambda d_1 \alpha_k^2 + d_3 \alpha_k^2 + i d_2 \alpha_k^3 = 0$$

is obtained.

The necessary condition of (SB) instability is

$$d_2 = d_3 = 0.$$

Then

$$\lambda_{1,k} = 0$$

and from (4.26)

$$\lambda_{2,3,k} = \pm \alpha_k \sqrt{d_1}.$$

While $d_1 > 0$ (see [3]) these are all real values and a half of them has positive signs, that is, this case is not a stability boundary.

For (DB) instability the necessary condition is

$$d_3 = 0.$$

Assume that $\lambda = \nu + i\omega$. Then at $\nu = 0$ for the imaginary parts from (4.26)

$$(4.27) \qquad -\omega^3 + \omega d_1 \alpha_k^2 + d_2 \alpha_k^3 = 0.$$

Equation (4.27) should always have at least one real solution $\omega = \omega(\alpha_k)$. For this kind of material, the dynamic type of instability (DB) is the only possible way of stability loss because the conditions of (SB) can only be satisfied in the instability region.

Now let us study the case when in (4.24) instead of ϵ the stress σ is present ($d_3 = 0$, $d_4 \neq 0$). Now the characteristic equation (3.7) with homogeneous boundary conditions is

$$(4.28) \qquad \lambda^3 - d_4 \lambda^2 + \lambda d_1 \alpha_k^2 + i d_2 \alpha_k^2 = 0.$$

The necessary condition of (SB) instability from (4.28) is

$$d_2 = 0$$

and for the other (nonzero) eigenvalues

(4.29) $$\lambda^2 - d_4\lambda + d_1\alpha_k^2 = 0.$$

The solutions of (4.29) are

(4.30) $$\lambda_{2,3,k} = \frac{d_4 \pm \sqrt{d_4^2 - 4\alpha_k^2 d_1}}{2}.$$

In (4.30) the sign of d_4 has a great importance. When it is negative, there is a static bifurcation, but in case of $d_4 > 0$ it does not exist.

For (DB) instability the bifurcation condition is

$$d_4 = 0$$

and then we have Eq. (4.27) and the same results as before for the eigenvalues.

5. Concluding remarks

The results show that by considering the system of the basic equations of a solid continuum as a dynamical system, the material stability conditions can be formulated by using the linear Lapunov stability conditions. The loss of stability can be classified into the classes of static and dynamic bifurcations. In one-dimensional mechanical problems the selection of the constitutive equation has the most important effect on the type of instability. We studied rate and gradient-dependent and independent constitutive equations as well. We found that rate-dependence in most of the cases separates the (SB) and (DB) types of instability, while the inclusion of gradient-dependence has a more complex effect. The one which was detected here is the role in determining the dimension of the critical eigenspace at the loss of stability. In case of the so-called second gradient-dependent material we have a very simple eigenspace at (SB), thus we could also perform a nonlinear static bifurcation investigation. For the other material models, the study ended up in infinite-dimensional critical nullspaces. Therefore an exciting theoretical question has arisen: does such behavior belong to the essence of the real physical phenomenon or it is caused only by the use of improper material models.

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