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Point force in a plane in the context of fractional nonlocal elasticity

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Solutions to the point load problems for elastic solids have different applications in geomechanics, contact mechanics, tribology as well as in modeling of lattice defects in crystals. Nonlocal elasticity assumes an integral constitutive equation for the stress tensor, takes into account interatomic long-range forces, reduces to the classical theory of elasticity in the long wave-length limit and to the atomic lattice theory in the short wave-length limit. Often, the nonlocal kernel of a stress constitutive equation is selected as the Green function of the Cauchy problem for appropriate partial differential equation. In this paper, we obtain the solution of elasticity problems for a point force in a plane and for a point load on the boundary of a half-plane in the context of the new theory of nonlocal elasticity in which the nonlocal modulus is the Green function of the Cauchy problem for the fractional diffusion equation with the Caputo derivative with respect to the nonlocality parameter and the fractional Riesz derivative with respect to spatial coordinates.

Key words: point force, Kelvin problem, Flamant problem, Cerruti problem, nonlocal elasticity, Caputo derivative, Riesz derivative.

1. Introduction

Solutions to the point load problems for elastic solids (see, for example, books [1–3]), on the one hand, play an important role as the Green functions for general loading. On the other hand, such solutions have different applications in geomechanics [1, 4–6], seismology [7], contact mechanics and tribology [8, 9] as well as in modeling of lattice defects in crystals [10–13].

The main disadvantage of classical elastic solutions consists in appearing of nonphysical singularities at the points of application of concentrated loads. In fact, the classical elastic solutions cannot describe the situation in the vicinity of the loading point. Moreover, the classical elasticity is not valid at this length scale and fails to explain the phenomena at the atomic level. This situation has given rise to the approaches which improve the corresponding classical elastic solutions.

In the semi-discrete approach [14–19], a portion of a crystal is treated as a discrete lattice, and for remainder of the solid an elastic continuum model is used. In the discrete region, the atoms are treated as classical particles, one deals with the individual displacements of the atoms. In the elastic region, displacements and stresses are calculated from the theory of elasticity. However, there is no prior knowledge of how large the discrete region should be. At the same time, the equations governing the behavior of the atoms in the discrete region increase rapidly in number and complexity as the volume of this region increases. In addition, the governing equations in the elastic region should be supplemented by the boundary conditions at the interface between the regions, and such conditions cannot be simply specified.

Great advances in improving the classical elasticity solutions have been made using nonlocal elasticity. The foundations of nonlocal theories, based on different schemes, were laid down by Eringen [20–24], Kunin [25], [26], and Podstrigach [27] (see also earlier investigations [28, 29] as well as the recent book [30], where additional references can be found).

Nonlocal elasticity takes into consideration interatomic long-range forces, gives rise to the classical theory of elasticity in the long wave-length limit and to the atomic lattice theory in the short wave-length limit. The stress at a reference point in the solid depends on the strains at all points of the body. The relation between the nonlocal stress tensor and the strain tensor has the nonlocal integral form. The constitutive equation for the nonlocal stress tensor $\mathbf{t}(\mathbf{x},\tau)$ is expressed as an integral of the classical stress tensor $\boldsymbol{\sigma}(\mathbf{x})$ with the weight function (nonlocality kernel) $K(\mathbf{x},\tau)$ [20–24]:

$$\mathbf{t}(\mathbf{x},\tau) = \int_V K(|\mathbf{x}-\mathbf{x}'|,\tau)\boldsymbol{\sigma}(\mathbf{x}')\,\mathrm{d}v(\mathbf{x}'). \tag{1.1}$$

Another version of the nonlocal theory was studied in [31, 32]. In this theory the nonlocal strain tensor $\mathbf{E}(\mathbf{x},\tau)$ is stated as an integral of classical strain tensor $\mathbf{e}(\mathbf{x},\tau)$ with the corresponding weight function

$$\mathbf{E}(\mathbf{x},\tau) = \int_V K(|\mathbf{x}-\mathbf{x}'|,\tau)\mathbf{e}(\mathbf{x}')\,\mathrm{d}v(\mathbf{x}'). \tag{1.2}$$

Starting from Eringen's investigations on nonlocal elasticity [22–24], the nonlocal kernels $K(|\mathbf{x}-\mathbf{x}'|,\tau)$ were chosen as the Green functions of the Cauchy problems of differential operators (see also subsequent papers [33–35]). It should be emphasized that the choice of different nonlocal kernels allows us to reflect various picularities of the problem.

A number of problems solved within the framework of the nonlocal theory of elasticity indicate its power. In particular, more justified results were obtained because the nonlocal elasticity is effective in removing nonphysical singularities occurring at points of applications of concentrated forces [36–40]. Another approach, which eliminates nonphysical singularities, involves the gradient elasticity [41, 42].

The fractional calculus has numerous applications in different branches of science (see, for example, [43–52] among many others). The fractional calculus was also used in various approaches to nonlocal theories [53–56]. The fractional gradient elasticity based on the fractional Laplacian in the Riesz form was studied in [57].

In the present paper, the theory elaborated in [58] is used to obtain nonlocal solutions for a point force in a plane (the Kelvin problem) and for a point load on the surface of a half-plane (the Flamant and Cerruti problems).

2. Governing equations

The nonlocality kernel $K(|\mathbf{x}-\mathbf{x}'|,\tau)$ appearing in Eqs. (1.1) and (1.2) includes the parameter τ associated with a characteristic length ratio l_0/l, where l_0 is an internal characteristic length and l is an external characteristic length [23, 24]. In the limit $\tau \to 0^+$, the kernel $K(|\mathbf{x}-\mathbf{x}'|,\tau)$ becomes the Dirac delta function

$$\lim_{\tau\to 0^+} K(|\mathbf{x}-\mathbf{x}'|,\tau) = \delta(|\mathbf{x}-\mathbf{x}'|). \tag{2.1}$$

ERINGEN [22, 24] introduced the nonlocal modulus $K(\mathbf{x},\tau)$ as the Green function of the Cauchy problem for the diffusion operator:

$$\frac{\partial K(\mathbf{x},\tau)}{\partial \tau} = a\,\Delta K(\mathbf{x},\tau), \tag{2.2}$$

$$\tau \to 0^+ : \quad K(\mathbf{x},\tau) = \delta(\mathbf{x}). \tag{2.3}$$

According to Eqs. (1.1) and (2.1), the associated Cauchy problem for the nonlocal stress tensor $\mathbf{t}(\mathbf{x},\tau)$ takes the form:

$$\frac{\partial \mathbf{t}(\mathbf{x},\tau)}{\partial \tau} = a\,\Delta \mathbf{t}(\mathbf{x},\tau), \tag{2.4}$$

$$\tau \to 0^+ : \quad \mathbf{t}(\mathbf{x},\tau) = \boldsymbol{\sigma}(\mathbf{x}). \tag{2.5}$$

In the fractional nonlocal elasticity proposed in [58], the nonlocal kernel $K(\mathbf{x},\tau)$ is introduced as the Green function of the Cauchy problem for the fractional diffusion equation:

$$\frac{\partial^\alpha K(\mathbf{x},\tau)}{\partial \tau^\alpha} = -a(-\Delta)^{\beta/2} K(\mathbf{x},\tau), \tag{2.6}$$

$$\tau \to 0^+ : \quad K(\mathbf{x},\tau) = \delta(\mathbf{x}), \tag{2.7}$$

under the assumption $0 < \alpha \le 1$ and $1 \le \beta \le 2$. The solution of the Cauchy problem (2.6) and (2.7) is expressed as

$$K(\mathbf{x},\tau) = \frac{1}{2\pi}\int_0^\infty E_\alpha(-\rho^\beta \tau^\alpha) J_0\big(\sqrt{x^2+y^2}\rho\big)\rho\,\mathrm{d}\rho, \tag{2.8}$$

where $E_\alpha(z)$ is the Mittag–Leffler function

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)}, \qquad \alpha > 0,\ z \in C. \tag{2.9}$$

In this case, the nonlocal stress tensor $\mathbf{t}(\mathbf{x},\tau)$ is found from the associated Cauchy problem:

$$\frac{\partial^\alpha \mathbf{t}(\mathbf{x},\tau)}{\partial \tau^\alpha} = -a(-\Delta)^{\beta/2} \mathbf{t}(\mathbf{x},\tau), \tag{2.10}$$

$$\tau \to 0^+ : \quad \mathbf{t}(\mathbf{x},\tau) = \boldsymbol{\sigma}(\mathbf{x}). \tag{2.11}$$

Here, $d^\alpha f(\tau)/d\tau^\alpha$ is the Caputo fractional derivative [59–61]

$$\frac{d^\alpha f(\tau)}{d\tau^\alpha} = \frac{1}{\Gamma(m-\alpha)}\int_0^\tau (\tau-u)^{m-\alpha-1}\frac{d^m f(u)}{du^m}\,\mathrm{d}u, \qquad m-1<\alpha<m. \tag{2.12}$$

The Laplace transform rule for the Caputo derivative (2.12) has the following form:

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(\tau)}{\mathrm{d}\tau^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)s^{\alpha-1-k}, \qquad m-1<\alpha<m, \tag{2.13}$$

where the asterisk marks the Laplace transform, s is the transform variable.

The Riesz derivative $(-\Delta)^{\beta/2} f(\mathbf{x})$ is the fractional generalization of the standard Laplace operator ($\beta = 2$) and is defined by its Fourier transform [59, 60]

$$\mathcal{F}\{(-\Delta)^{\beta/2} f(\mathbf{x})\} = |\boldsymbol{\xi}|^\beta \mathcal{F}\{f(\mathbf{x})\}, \qquad \beta > 0. \tag{2.14}$$

Here, $\mathbf{x}$ is a vector of space variables, $\boldsymbol{\xi}$ is a vector of Fourier transform variables.

It should be emphasized that though the parameter τ in Eqs. (2.2), (2.4), (2.6), and (2.10) looks like time in the corresponding diffusion equation, in fact, it has quite different physical meaning: this is the parameter describing the spatial nonlocality.

3. Point force in a plane

3.1. Kelvin problem

In the Cartesian coordinates (x, y), the concentrated force in an elastic plane is modeled by the body force (F_x, F_y) in the equilibrium equation as:

$$F_x = F_1\,\delta(x)\,\delta(y), \qquad F_y = F_2\,\delta(x)\,\delta(y). \tag{3.1}$$

The classical elasticity solution for the plane Kelvin problem can be found in the books on the theory of elasticity (see, for example, [1–3]) and in Cartesian coordinates has the following form for the force acting in the y-direction:

$$\sigma_{xx} = -\frac{F_2(1+\nu)}{4\pi}\,\frac{y}{x^2+y^2}\left(\frac{2\nu}{1+\nu} + \frac{x^2-y^2}{x^2+y^2}\right), \tag{3.2}$$

$$\sigma_{yy} = -\frac{F_2(1+\nu)}{4\pi}\,\frac{y}{x^2+y^2}\left(\frac{2}{1+\nu} - \frac{x^2-y^2}{x^2+y^2}\right), \tag{3.3}$$

$$\sigma_{xy} = -\frac{F_2(1+\nu)}{4\pi}\,\frac{x}{x^2+y^2}\left(\frac{1-\nu}{1+\nu} + \frac{2y^2}{x^2+y^2}\right), \tag{3.4}$$

where ν is the Poisson ratio.

In this case, according to (2.10) and (2.11), we have the following Cauchy problems for the components of the nonlocal stress tensor:

$$\frac{\partial^\alpha t_{xx}}{\partial \tau^\alpha} = -a(-\Delta)^{\beta/2}\,t_{xx}, \tag{3.5}$$

$$\tau = 0: \quad t_{xx} = -\frac{F_2(1+\nu)}{4\pi}\,\frac{y}{x^2+y^2}\left(\frac{2\nu}{1+\nu} + \frac{x^2-y^2}{x^2+y^2}\right), \tag{3.6}$$

$$\frac{\partial^\alpha t_{yy}}{\partial \tau^\alpha} = -a(-\Delta)^{\beta/2}\,t_{yy}, \tag{3.7}$$

$$\tau = 0: \quad t_{yy} = -\frac{F_2(1+\nu)}{4\pi}\,\frac{y}{x^2+y^2}\left(\frac{2}{1+\nu} - \frac{x^2-y^2}{x^2+y^2}\right), \tag{3.8}$$

$$\frac{\partial^\alpha t_{xy}}{\partial \tau^\alpha} = -a(-\Delta)^{\beta/2}\,t_{xy}, \tag{3.9}$$

$$\tau = 0: \quad t_{xy} = -\frac{F_2(1+\nu)}{4\pi}\,\frac{x}{x^2+y^2}\left(\frac{1-\nu}{1+\nu} + \frac{2y^2}{x^2+y^2}\right). \tag{3.10}$$

The Laplace transform with respect to the nonlocality parameter τ and the Fourier transform with respect to the spatial coordinates x and y after using integrals (A.1)–(A.6) from Appendix give:

$$\widetilde{t}^{\,*}_{xx} = -\frac{F_2(1+\nu)\,i}{2\pi}\,\frac{s^{\alpha-1}}{s^\alpha + a(\xi^2+\eta^2)^{\beta/2}}\left[\frac{\nu}{1+\nu}\,\frac{\eta}{\xi^2+\eta^2} - \frac{\eta\,\xi^2}{(\xi^2+\eta^2)^2}\right], \tag{3.11}$$

$$\widetilde{t}^{*}_{yy} = -\frac{F_2(1+\nu)\,i}{2\pi}\,\frac{s^{\alpha-1}}{s^{\alpha}+a(\xi^2+\eta^2)^{\beta/2}}\left[\frac{1}{1+\nu}\,\frac{\eta}{\xi^2+\eta^2}+\frac{\eta\,\xi^2}{(\xi^2+\eta^2)^2}\right], \tag{3.12}$$

$$\widetilde{t}^{*}_{xy} = -\frac{F_2(1+\nu)\,i}{2\pi}\,\frac{s^{\alpha-1}}{s^{\alpha}+a(\xi^2+\eta^2)^{\beta/2}}\left[\frac{1}{1+\nu}\,\frac{\xi}{\xi^2+\eta^2}-\frac{\xi\,\eta^2}{(\xi^2+\eta^2)^2}\right]. \tag{3.13}$$

Inverting the integral transforms taking into account the following formula [60, 61]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+b}\right\} = E_{\alpha}(-b\,\tau^{\alpha}), \tag{3.14}$$

where $E_{\alpha}(z)$ is the Mittag–Leffler function (2.9), we get:

$$t_{xx} = -\frac{F_2(1+\nu)}{\pi^2}\int_0^{\infty}\int_0^{\infty} E_{\alpha}[-a(\xi^2+\eta^2)^{\beta/2}\tau^{\alpha}] \times\left[\frac{\nu}{1+\nu}\,\frac{\eta}{\xi^2+\eta^2}-\frac{\eta\,\xi^2}{(\xi^2+\eta^2)^2}\right]\cos(x\xi)\sin(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{3.15}$$

$$t_{yy} = -\frac{F_2(1+\nu)}{\pi^2}\int_0^{\infty}\int_0^{\infty} E_{\alpha}[-a(\xi^2+\eta^2)^{\beta/2}\tau^{\alpha}] \times\left[\frac{1}{1+\nu}\,\frac{\eta}{\xi^2+\eta^2}+\frac{\eta\,\xi^2}{(\xi^2+\eta^2)^2}\right]\cos(x\xi)\sin(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{3.16}$$

$$t_{xy} = -\frac{F_2(1+\nu)}{\pi^2}\int_0^{\infty}\int_0^{\infty} E_{\alpha}[-a(\xi^2+\eta^2)^{\beta/2}\tau^{\alpha}] \times\left[\frac{1}{1+\nu}\,\frac{\xi}{\xi^2+\eta^2}-\frac{\xi\,\eta^2}{(\xi^2+\eta^2)^2}\right]\sin(x\xi)\cos(y\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{3.17}$$

Next we introduce the polar coordinate system in the (ξ,η)-plane and after some mathematical transformations, using integrals (A.7) and (A.8) from Appendix, we finally obtain:

$$t_{xx} = -\frac{F_2(1+\nu)}{8\pi}\int_0^{\infty} E_{\alpha}(-a\rho^{\beta}\tau^{\alpha})\left[\frac{3\nu-1}{1+\nu}\,\frac{y}{\sqrt{x^2+y^2}}\,J_1(\rho\sqrt{x^2+y^2}) - \frac{y(y^2-3x^2)}{(x^2+y^2)^{3/2}}\,J_3\big(\rho\sqrt{x^2+y^2}\big)\right]\mathrm{d}\rho, \tag{3.18}$$

$$t_{yy} = -\frac{F_2(1+\nu)}{8\pi}\int_0^{\infty} E_{\alpha}(-a\rho^{\beta}\tau^{\alpha})\left[\frac{5+\nu}{1+\nu}\,\frac{y}{\sqrt{x^2+y^2}}\,J_1\big(\rho\sqrt{x^2+y^2}\big) + \frac{y(y^2-3x^2)}{(x^2+y^2)^{3/2}}\,J_3\big(\rho\sqrt{x^2+y^2}\big)\right]\mathrm{d}\rho, \tag{3.19}$$

$$t_{xy} = -\frac{F_2(1+\nu)}{8\pi} \int_0^\infty E_\alpha(-a\rho^\beta \tau^\alpha) \left[\frac{3-\nu}{1+\nu} \frac{x}{\sqrt{x^2+y^2}} J_1(\rho\sqrt{x^2+y^2}) - \frac{x(x^2-3y^2)}{(x^2+y^2)^{3/2}} J_3(\rho\sqrt{x^2+y^2}) \right] \mathrm{d}\rho. \tag{3.20}$$

Now we consider several particular cases of the obtained solution. The standard diffusion equation for the nonlocality kernel corresponds to the values $\alpha = 1$, $\beta = 2$. Then $E_1(-z) = \mathrm{e}^{-z}$, and Eqs. (A.9) and (A.10) from Appendix give:

$$t_{xx} = -\frac{F_2(1+\nu)}{8\pi} \left\{ \frac{3\nu-1}{1+\nu} \frac{y}{x^2+y^2} \left[1-\exp\left(-\frac{x^2+y^2}{4a\tau}\right) \right] - \frac{y(y^2-3x^2)}{(x^2+y^2)^2} \left[1-\frac{8a\tau}{x^2+y^2} + \left(1+\frac{8a\tau}{x^2+y^2}\right) \exp\left(-\frac{x^2+y^2}{4a\tau}\right) \right] \right\}, \tag{3.21}$$

$$t_{yy} = -\frac{F_2(1+\nu)}{8\pi} \left\{ \frac{5+\nu}{1+\nu} \frac{y}{x^2+y^2} \left[1-\exp\left(-\frac{x^2+y^2}{4a\tau}\right) \right] + \frac{y(y^2-3x^2)}{(x^2+y^2)^2} \left[1-\frac{8a\tau}{x^2+y^2} + \left(1+\frac{8a\tau}{x^2+y^2}\right) \exp\left(-\frac{x^2+y^2}{4a\tau}\right) \right] \right\}, \tag{3.22}$$

$$t_{xy} = -\frac{F_2(1+\nu)}{8\pi} \left\{ \frac{3-\nu}{1+\nu} \frac{x}{x^2+y^2} \left[1-\exp\left(-\frac{x^2+y^2}{4a\tau}\right) \right] - \frac{x(x^2-3y^2)}{(x^2+y^2)^2} \left[1-\frac{8a\tau}{x^2+y^2} + \left(1+\frac{8a\tau}{x^2+y^2}\right) \exp\left(-\frac{x^2+y^2}{4a\tau}\right) \right] \right\}. \tag{3.23}$$

In the case of the Cauchy diffusion equation for the nonlocality kernel ($\alpha = 1$, $\beta = 1$), using integrals (A.11) and (A.12) from Appendix, we arrive at:

$$t_{xx} = -\frac{F_2(1+\nu)}{8\pi} \frac{y}{\sqrt{a^2\tau^2+x^2+y^2}\,(a\tau+\sqrt{a^2\tau^2+x^2+y^2})} \times \left[\frac{3\nu-1}{1+\nu} - \frac{y^2-3x^2}{(a\tau+\sqrt{a^2\tau^2+x^2+y^2})^2} \right], \tag{3.24}$$

$$t_{yy} = -\frac{F_2(1+\nu)}{8\pi} \frac{y}{\sqrt{a^2\tau^2+x^2+y^2}\,(a\tau+\sqrt{a^2\tau^2+x^2+y^2})} \times \left[\frac{5+\nu}{1+\nu} + \frac{y^2-3x^2}{(a\tau+\sqrt{a^2\tau^2+x^2+y^2})^2} \right], \tag{3.25}$$

$$t_{xy} = -\frac{F_2(1+\nu)}{8\pi} \frac{x}{\sqrt{a^2\tau^2+x^2+y^2}\,(a\tau+\sqrt{a^2\tau^2+x^2+y^2})} \times \left[\frac{3-\nu}{1+\nu} - \frac{x^2-3y^2}{(a\tau+\sqrt{a^2\tau^2+x^2+y^2})^2} \right]. \tag{3.26}$$

Another particular case corresponds to the values $\alpha \to 0$, $\beta = 2$ when the Mittag–Leffler function tends to $E_0(-z) = 1/(1+z)$. Integrals (A.13) and (A.14) from Appendix allow us to evaluate the components of the nonlocal stress tensor:

$$t_{xx} = -\frac{F_2(1+\nu)}{8\pi}\left\{\frac{3\nu-1}{1+\nu}\,\frac{y}{x^2+y^2}\left[1-\sqrt{\frac{x^2+y^2}{a}}\,K_1\left(\sqrt{\frac{x^2+y^2}{a}}\right)\right]\right.$$
$$\left.-\frac{y(y^2-3x^2)}{(x^2+y^2)^2}\left[1-\frac{8a}{x^2+y^2}+\sqrt{\frac{x^2+y^2}{a}}\,K_3\left(\sqrt{\frac{x^2+y^2}{a}}\right)\right]\right\}, \tag{3.27}$$

$$t_{yy} = -\frac{F_2(1+\nu)}{8\pi}\left\{\frac{5+\nu}{1+\nu}\,\frac{y}{x^2+y^2}\left[1-\sqrt{\frac{x^2+y^2}{a}}\,K_1\left(\sqrt{\frac{x^2+y^2}{a}}\right)\right]\right.$$
$$\left.+\frac{y(y^2-3x^2)}{(x^2+y^2)^2}\left[1-\frac{8a}{x^2+y^2}+\sqrt{\frac{x^2+y^2}{a}}\,K_3\left(\sqrt{\frac{x^2+y^2}{a}}\right)\right]\right\}, \tag{3.28}$$

$$t_{xy} = -\frac{F_2(1+\nu)}{8\pi}\left\{\frac{3-\nu}{1+\nu}\,\frac{x}{x^2+y^2}\left[1-\sqrt{\frac{x^2+y^2}{a}}\,K_1\left(\sqrt{\frac{x^2+y^2}{a}}\right)\right]\right.$$
$$\left.-\frac{x(x^2-3y^2)}{(x^2+y^2)^2}\left[1-\frac{8a}{x^2+y^2}+\sqrt{\frac{x^2+y^2}{a}}\,K_3\left(\sqrt{\frac{x^2+y^2}{a}}\right)\right]\right\}. \tag{3.29}$$

Let l_0 be a characteristic length of the medium. We introduce the following nondimensional quantities:

$$\bar{x} = \frac{x}{l_0}, \qquad \bar{y} = \frac{y}{l_0}, \qquad \bar{\tau} = \frac{a^{1/\alpha}}{l_0^{\beta/\alpha}}\,\tau, \qquad \bar{t}_{xx} = \frac{4\pi l_0}{F_2}\,t_{xx}. \tag{3.30}$$

Figures present the dependence of the stress component $\bar{t}_{xx}$ on distance $\bar{y}$ for $\bar{x} = 0$ and typical values of orders of fractional operators and of a nondimensional parameter. In numerical calculations, we have taken $\nu = 0.25$.

The maximum values of the stress component and the coordinates of points at which they are achieved are presented in Table 1 depending on the orders of derivatives α and β and on the nonlocality parameter $\bar{\tau}$. It is obvious from Figs. 1–3 and Table 1 that nonlocal solutions do not have nonphysical singularities. The approximate estimates of maximum values of nonlocal stress are quite acceptable from a physical point of view and are comparable with the similar estimates for concentrated forces carried out in [36, 37, 39] and in other problems solved in the framework of nonlocal elasticity [23, 24].

When $\beta = 2$, the problem can be studied in polar coordinates (r, θ). In this case, the solution of the classical two-dimensional Kelvin problem is written as [2]:

$$\sigma_{rr} = -\frac{(3+\nu)}{4\pi r} F_2 \sin\theta, \tag{3.31}$$

$$\sigma_{\theta\theta} = \frac{(1-\nu)}{4\pi r} F_2 \sin\theta, \tag{3.32}$$

$$\sigma_{r\theta} = -\frac{(1-\nu)}{4\pi r} F_2 \cos\theta. \tag{3.33}$$

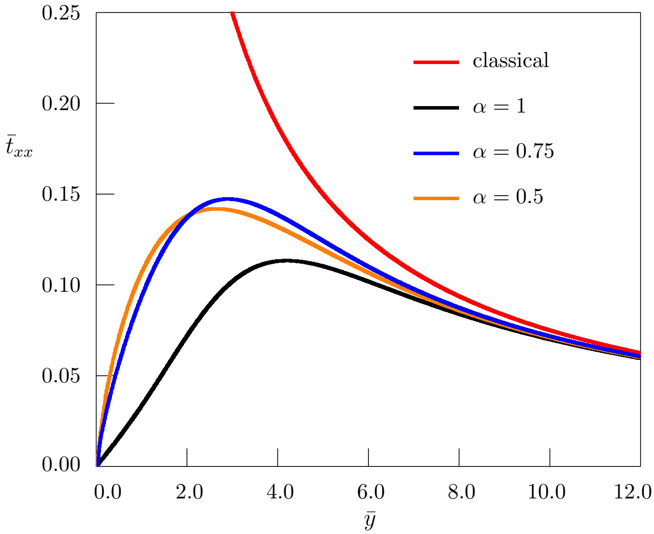

FIG. 1. Dependence of nonlocal stress component $\bar{t}_{xx}$ on distance $\bar{y}$ for different values of the order of fractional derivative α ($\beta = 2$).

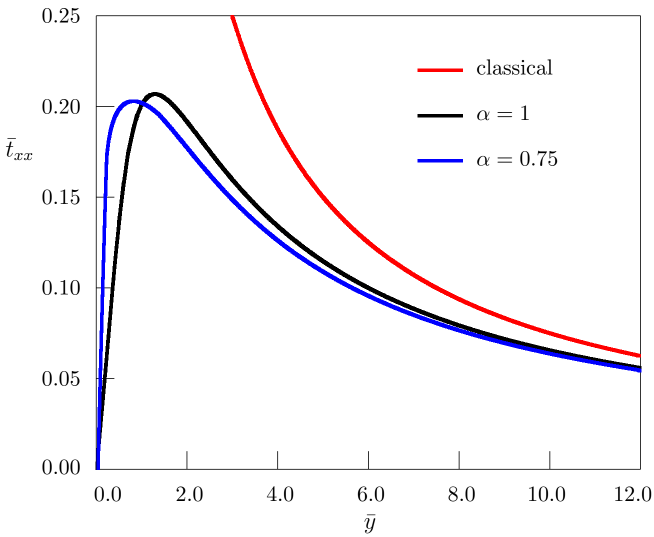

FIG. 2. Dependence of nonlocal stress component $\bar{t}_{xx}$ on distance $\bar{y}$ for different values of the order of fractional derivative α ($\beta = 1$).

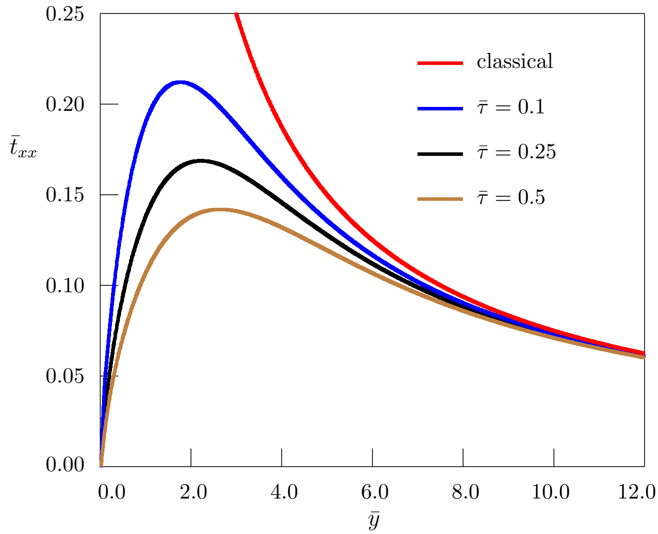

FIG. 3. Dependence of nonlocal stress component $\bar{t}_{xx}$ on distance $\bar{y}$ for different values of the nonlocality parameter $\bar{\tau}$ ($\alpha = 0.5$, $\beta = 2$).

TABLE 1. The maximum values of the stress component.

α	β	$\bar{\tau}$	$\bar{y}^{\max}$	$\bar{t}_{xx}^{\max}$
0.75	1	0.5	0.8	0.204
0.85	1	0.5	1.1	0.200
1	1	0.5	1.3	0.207
1	2	0.5	4.2	0.113
0.75	2	0.5	2.9	0.147
0.5	2	0.5	2.6	0.142
0.5	2	0.25	2.2	0.169
0.5	2	0.1	1.8	0.212

The method of solving fractional nonlocal elasticity problems in polar coordinates was proposed in [58]. This method is based on representation of the components of the Laplacian of the stress tensor $\mathbf{t}(r, \theta)$ in polar coordinates [62] and splitting a system of couple equations by introducing the auxiliary functions (for details see Appendix B and [58]). Below we present the solution of the nonlocal two-dimensional Kelvin problem in polar coordinates (the result corresponding to the solution (3.18)–(3.20) for $\beta = 2$):

$$t_{rr} = -\frac{F_2}{8\pi} \int_0^\infty E_\alpha(-a\xi^2\tau^\alpha)\big[(5+\nu)J_1(r\xi) + (1+\nu)J_3(r\xi)\big]\,\mathrm{d}\xi \sin\theta, \tag{3.34}$$

$$(3.35)\qquad t_{\theta\theta} = \frac{F_2}{8\pi}\int_0^\infty E_\alpha(-a\xi^2\tau^\alpha)\big[(1-3\nu)J_1(r\xi)+(1+\nu)J_3(r\xi)\big]\,\mathrm{d}\xi\sin\theta,$$

$$(3.36)\qquad t_{r\theta} = -\frac{F_2}{8\pi}\int_0^\infty E_\alpha(-a\xi^2\tau^\alpha)\big[(3-\nu)J_1(r\xi)-(1+\nu)J_3(r\xi)\big]\,\mathrm{d}\xi\cos\theta.$$

3.2. Flamant problem and Cerruti problem

The classical elasticity solution for a point force applied at the origin of the coordinates normally to the boundary of a half-plane (the Flamant problem) has the following form [2]:

$$(3.37)\qquad \sigma_{rr} = -\frac{2F_2}{\pi r}\sin\theta, \qquad \sigma_{\theta\theta}=0, \qquad \sigma_{r\theta}=0.$$

In this case, the solution of the nonlocal two-dimensional Flamant problem reads:

$$(3.38)\qquad t_{rr} = -\frac{F_2}{2\pi}\int_0^\infty E_\alpha(-a\xi^2\tau^\alpha)\big[3J_1(r\xi)+J_3(r\xi)\big]\,\mathrm{d}\xi\sin\theta,$$

$$(3.39)\qquad t_{\theta\theta} = -\frac{F_2}{2\pi}\int_0^\infty E_\alpha(-a\xi^2\tau^\alpha)\big[J_1(r\xi)-J_3(r\xi)\big]\,\mathrm{d}\xi\sin\theta,$$

$$(3.40)\qquad t_{r\theta} = -\frac{F_2}{2\pi}\int_0^\infty E_\alpha(-a\xi^2\tau^\alpha)\big[J_1(r\xi)-J_3(r\xi)\big]\,\mathrm{d}\xi\cos\theta.$$

For the point force applied at the origin of the coordinates tangentially to the boundary of a half-plane (the Cerruti problem), the components of the classical stress tensor are [2]:

$$(3.41)\qquad \sigma_{rr} = -\frac{2F_1}{\pi r}\cos\theta, \qquad \sigma_{\theta\theta}=0, \qquad \sigma_{r\theta}=0.$$

Comparison of Eqs. (3.37) and (3.41) taking into account Eqs. (B.1)–(B.3) from Appendix B reveals that the solution of the nonlocal two-dimensional Cerruti problem is expressed by the solution of the Flamant problem (Eqs. (3.38)–(3.40)) with F_2 substituted by F_1 as well as $\sin\theta$ substituted by $\cos\theta$ and vice versa.

Though for the classical solutions of the Flamant and Cerruti problems $\sigma_{\theta\theta}=0$, $\sigma_{r\theta}=0$, the interesting feature of the nonlocal solution for these problems is that $t_{\theta\theta}\neq 0$, $t_{r\theta}\neq 0$ and only for the vanishing nonlocality parameter:

$$(3.42)\qquad \lim_{\tau\to 0^+} t_{\theta\theta} = \sigma_{\theta\theta} = 0, \qquad \lim_{\tau\to 0^+} t_{r\theta} = \sigma_{r\theta} = 0.$$

4. Concluding remarks

We have solved the problem of a concentrated force in an elastic plane using the fractional nonlocal elasticity theory with the nonlocality kernel being the fundamental solution to the generalized diffusion equation with the Caputo derivative of the order $0 < \alpha \leq 1$ with respect to the parameter describing nonlocality and the Riesz derivative of the order $1 \leq \beta \leq 2$ with respect to the spatial coordinates. The unbounded stress obtained in the solutions of the corresponding problems of the classical therory of elasticity is a great obstacle in interpretation of results and in their applications. The main advantage of the nonlocal theory consists in elimination of nonphysical singularities of solutions. The solutions obtained within the framework of nonlocal elasticity give the finite values of stresses which maximum is achived at some distance from the point of application of a concentrated load and is quite adequate from a physical point of view. The point of the maximum and the maximum value of stress depend on the orders of the Caputo and Riesz derivatives.

The solutions of the considered problems were obtained in terms of integrals with integrands containing the Mittag–Leffler function $E_\alpha(z)$. To evaluate the Mittag–Leffler function, the algorithm proposed in [63] was used.

Fractional nonlocal elasticity can be useful for better matching the classical theory of elasticity and the atomic lattice theory as well as for the unified description of processes at micro-, meso- and macro-levels.

In the future studies, we will extend the obtained results to concentrated forces in the three-dimensional solid and to other problems of classical elasticity to eliminate unbounded stresses.

Appendix A

The following integrals are used in the paper. The majority of integrals are taken from the books by Prudnikov, Brychkov and Marichev [64, 65]; integrals (A.8), (A.10), (A.13), and (A.14) have been evaluated by the authors. In all the equations it is assumed that $p > 0$, $q > 0$,

$$\int_0^\infty \frac{1}{x^2+p^2}\cos(qx)\,\mathrm{d}x = \frac{\pi}{2p}\,\mathrm{e}^{-pq}, \tag{A.1}$$

$$\int_0^\infty \frac{x}{x^2+p^2}\sin(qx)\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{e}^{-pq}, \tag{A.2}$$

$$\int_0^\infty \mathrm{e}^{-px}\cos(qx)\,\mathrm{d}x = \frac{p}{p^2+q^2}, \tag{A.3}$$

$$\int_0^\infty \mathrm{e}^{-px}\sin(qx)\,\mathrm{d}x = \frac{q}{p^2+q^2}, \tag{A.4}$$

$$\int_0^\infty \frac{x^2}{(x^2+p^2)^2}\cos(qx)\,\mathrm{d}x = \frac{\pi}{4}\,\frac{1-pq}{p}\,\mathrm{e}^{-pq}, \tag{A.5}$$

$$\int_0^\infty x\,\mathrm{e}^{-px}\sin(qx)\,\mathrm{d}x = \frac{2pq}{(p^2+q^2)^2}, \tag{A.6}$$

$$\int_0^1 \sin(p\sqrt{1-x^2})\cos(qx)\,\mathrm{d}x = \frac{\pi}{2}\,\frac{p}{\sqrt{p^2+q^2}}\,J_1(\sqrt{p^2+q^2}), \tag{A.7}$$

$$\int_0^1 x^2\,\sin(p\sqrt{1-x^2})\cos(qx)\,\mathrm{d}x \tag{A.8}$$

$$= \frac{\pi}{8}\,\frac{p}{\sqrt{p^2+q^2}}\left[J_1(\sqrt{p^2+q^2}) + \frac{p^2-3q^2}{p^2+q^2}\,J_3(\sqrt{p^2+q^2})\right],$$

$$\int_0^\infty e^{-px^2}\,J_1(qx)\,\mathrm{d}x = \frac{1}{q}\left[1-\exp\left(-\frac{q^2}{4p}\right)\right], \tag{A.9}$$

$$\int_0^\infty e^{-px^2}\,J_3(qx)\,\mathrm{d}x = \frac{1}{q}\left[1-\frac{8p}{q^2}+\left(1+\frac{8p}{q^2}\right)\exp\left(-\frac{q^2}{4p}\right)\right], \tag{A.10}$$

$$\int_0^\infty e^{-px}\,J_1(qx)\,\mathrm{d}x = \frac{q}{\sqrt{p^2+q^2}(p+\sqrt{p^2+q^2})}, \tag{A.11}$$

$$\int_0^\infty e^{-px}\,J_3(qx)\,\mathrm{d}x = \frac{q^3}{\sqrt{p^2+q^2}(p+\sqrt{p^2+q^2})^3}, \tag{A.12}$$

$$\int_0^\infty \frac{1}{x^2+p^2}\,J_1(qx)\,\mathrm{d}x = \frac{1}{qp^2}-\frac{1}{p}\,K_1(pq), \tag{A.13}$$

$$\int_0^\infty \frac{1}{x^2+p^2}\,J_3(qx)\,\mathrm{d}x = \frac{1}{qp^2}-\frac{8}{q^3p^4}+\frac{1}{p}\,K_3(pq), \tag{A.14}$$

where $J_n(z)$ is the Bessel function of the first kind, $K_n(z)$ is the modified Bessel function.

Appendix B

The components of the Laplacian of the symmetric tensor in polar coordinates were derived in [62] and are expressed as:

$$(\Delta t)_{rr} = \Delta t_{rr} - \frac{4}{r^2}\frac{\partial t_{r\theta}}{\partial \theta} - \frac{2}{r^2}(t_{rr} - t_{\theta\theta}), \tag{B.1}$$

$$(\Delta t)_{\theta\theta} = \Delta t_{\theta\theta} + \frac{4}{r^2}\frac{\partial t_{r\theta}}{\partial \theta} + \frac{2}{r^2}(t_{rr} - t_{\theta\theta}), \tag{B.2}$$

$$(\Delta t)_{r\theta} = \Delta t_{r\theta} - \frac{4}{r^2}t_{r\theta} + \frac{2}{r^2}\frac{\partial (t_{rr} - t_{\theta\theta})}{\partial \theta}, \tag{B.3}$$

where

$$\Delta t_{ij} = \frac{\partial^2 t_{ij}}{\partial r^2} + \frac{1}{r}\frac{\partial t_{ij}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t_{ij}}{\partial \theta^2}. \tag{B.4}$$

If the components of the stress tensor are of the form:

$$t_{rr}(r,\theta) = T_{rr}(r)\sin\theta,\ t_{\theta\theta}(r,\theta) = T_{\theta\theta}(r)\sin\theta,\ t_{r\theta}(r,\theta) = T_{r\theta}(r)\cos\theta, \tag{B.5}$$

when the system of coupled equations (B.1)–(B.3) can be splitted into independent equations by introducing the new functions:

$$f(r) = T_{rr}(r) + T_{\theta\theta}(r), \tag{B.6}$$

$$g(r) = T_{rr}(r) - T_{\theta\theta}(r) + 2T_{r\theta}(r), \tag{B.7}$$

$$h(r) = T_{rr}(r) - T_{\theta\theta}(r) - 2T_{r\theta}(r). \tag{B.8}$$

References

1. J.R. Barber, *Elasticity*, 2nd ed., Kluwer Academic Publishers, Dordrecht, 2002.
2. M. Kachanov, B. Shafiro, I. Tsukrov, *Handbook of Elasticity Solutions*, Springer, Dordrecht, 2003.
3. S.P. Timoshenko, J.N. Goodier, *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1970.
4. A.F. Bower, *Applied Mechanics of Solids*, CRC Press, Boca Raton, 2010.
5. R.O. Davis, A.P.S. Selvadurai, *Elasticity and Geomechanics*, Cambridge University Press, Cambridge, 1996.
6. A.P.S. Selvadurai, *The analytical method in geomechanics*, Applied Mechanics Reviews, **60**, 87–106, 2007.
7. M. Båth, *Mathematical Aspects of Seismology*, Elsevier, Amsterdam, 1968.
8. J.R. Barber, *Contact Mechanics*, Springer, Cham, 2018.

9. G.M.L. Gladwell, *Contact Problems in the Classical Theory of Elasticity*, Springer, Dordrecht, 1980.

10. R.W. Balluffi, *Introduction to Elasticity Theory for Crystal Defects*, 2nd ed., World Scientific, Singapore, 2016.

11. R. Phillips, *Crystal, Defects and Microstructures: Modeling Across Scales*, Cambridge University Press, Cambridge, 2001.

12. J.P. Hirth, J. Lothe, *Theory of Dislocations*, 2nd ed., Wiley, New York, 1982.

13. C. Teodosiu, *Elastic Models of Crystal Defects*, Springer, Berlin, 1982.

14. L. Tewordt, *Distortion of the lattice around an interstitial, a crowdion, and a vacancy in copper*, Physical Review, **109**, 61–68, 1958.

15. W.-M. Shyu, D. Brust, F.G. Fumi, *Relaxation effects around vacances in sodoium metal*, Journal of Physics and Chemistry of Solids, **28**, 717–724, 1967.

16. A. Lidiard, *The Mott–Littleton method: an introductory survey*, Journal of the Chemical Society, Faraday Transactions 2: Molecular and Chemical Physics, **85**, 341–349, 1982.

17. C.R.A. Catlow [ed.], *Computer Modelling in Inorganic Crystallography*, Academic Press, London, 1997.

18. M.S. Duesbery, *Modeling of the dislocation core*, [in:] Dislocations in Solids, vol. 8: Basic Problems and Applications, North-Holland, Amsterdam, pp. 67–173, 1989.

19. M.S. Duesbery, G.Y. Richardson, *The dislocation core in crystalline materials*, Critical Reviews in Solid State and Materials Sciences, **17**, 1–46, 1991.

20. A.C. Eringen, *Linear theory of nonlocal elasticity and dispersion of plane waves*, International Journal of Engineering Science, **10**, 425–435, 1972.

21. A.C. Eringen, *Nonlocal polar field theories*, [in:] Continuum Physics, vol. 4, Polar and Nonlocal Field Theories, Academic Press, New York, pp. 205–267, 1976.

22. A.C. Eringen, *On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves*, Journal of Applied Physics, **54**, 4703–4710, 1983.

23. A.C. Eringen, *Vistas on nonlocal continuum physics*, International Journal of Engineering Science, **30**, 1551–1565, 1992.

24. A.C. Eringen, *Nonlocal Continuum Field Theories*, Springer, New York, 2002.

25. I.A. Kunin, *Elastic Media with Microstructure I. One-Dimensional Models*, Springer, Berlin, 1982.

26. I.A. Kunin, *Elastic Media with Microstructure II. Three-Dimensional Models*, Springer, Berlin, 1983.

27. Ya.S. Podstrigach, *On a nonlocal theory of solid body deformation*, Soviet Applied Mechanics, **3**, 2, 44–46, 1967.

28. E. Kröner, *Elasticity theory of materials with long range cohesive forces*, International Journal of Solids and Structures, **3**, 731–742, 1967.

29. J.A. Krumhansl, *Some considerations of the relation between solid state physics and generalized continuum mechanics*, [in:] Mechanics of Generalized Continua, Springer, Berlin, pp. 298–311, 1968.

30. E. Ghavanloo, H. Rafii-Tabar, S.A. Fazelzadeh, *Computational Continuum Mechanics of Nanoscopic Structures: Nonlocal Elasticity Approaches*, Springer, Cham, 2019.

31. E. Benvenuti, A. Simone, *One-dimensional nonlocal and gradient elasticity: Closed-form solution and size effect*, **48**, 46–51, 2013.

32. G. Romano, R. Barretta, *Stress-driven versus strain-driven nonlocal integral model for elastic nano-beams*, Composites Part B, **114**, 184–188, 2017.

33. A. Apuzzo, R. Barretta, S.A. Faghidian, R. Luciano, F. Marotti De Sciarra, *Free vibrations of elastic beams by modified nonlocal strain gradient theory*, International Journal of Engineering Science, **133**, 99–108, 2018.

34. M. Lazar, G.A. Maugin, E.C. Aifantis, *On a theory of nonlocal elasticity of bi-Helmholtz type and some applications*, International Journal of Solids and Structures, **43**, 1404–1421, 2006.

35. G. Romano, R. Barretta, *Nonlocal elasticity in nanobeams: the stress-driven integral model*, International Journal of Engineering Science, **115**, 14–27, 2017.

36. R. Artan, *Nonlocal elastic half plane loaded by a concentrated force*, International Journal of Engineering Science, **34**, 943–950, 1996.

37. R. Artan, The nonlocal solution of the elastic half-plane loaded at the origin by shear force, Turkish Journal of Engineering and Environmental Sciences, **25**, 391–403, 2001.

38. J.L. Nowinski, *On a three-dimensional Kelvin problem for an elastic nonlocal medium*, Acta Mechanica, **84**, 77–87, 1990.

39. Y. Povstenko, I. Kubik, *Concentrated ring loading in a nonlocal elastic medium*, International Journal of Engineering Science, **43**, 457–471, 2005.

40. R. Wang, *Anti-plane line force in nonlocal elasticity*, Mechanics Research Communications, **16**, 307–309, 1989.

41. M. Lazar, G.A. Maugin, *A note on line forces in gradient elasticity*, Mechanics Research Communications, **33**, 674–680, 2006.

42. S. Li, I. Miskioglu, B.S. Altan, *Solution to line loading of a semi-infinite solid in gradient elasticity*, International Journal of Solids and Structures, **41**, 3395–3410, 2004.

43. T.M. Atanacković, S. Pilipović, B. Stanković, D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes*, John Wiley & Sons, Hoboken, 2014.

44. R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, 2nd ed., World Scientific, Singapore, 2014.

45. R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, Redding, 2006.

46. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press, London, 2010.

47. Y. Povstenko, *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*, Birkhäuser, New York, 2015.

48. Y. Povstenko, *Fractional Thermoelasticity*, 2nd ed., Springer, Cham, 2024.

49. V.E. TARASOV, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Berlin, 2010.

50. V.E. TARASOV, V.V. TARASOVA, *Economic Dynamics with Memory: Fractional Calculus Approach*, De Gruyter, Berlin, 2021.

51. V.V. UCHAIKIN, *Fractional Derivatives for Physicists and Engineers, Vol. I: Background and Theory*, Springer, Berlin and Higher Education Press, Beijing, 2013.

52. V.V. UCHAIKIN, *Fractional Derivatives for Physicists and Engineers, Vol. II: Applications*, Springer, Berlin and Higher Education Press, Beijing, 2013.

53. M. DI PAOLA, M. ZINGALES, *Long-range cohesive interactions of non-local continuum faced by fractional calculus*, International Journal of Solids and Structures, **45**, 5642–5659, 2008.

54. N. CHALLAMEL, D. ZORICA, T.M. ATANACKOVIĆ, D.T. SPASIĆ, *On the fractional generalization of Eringen's nonlocal elasticity for wave propagation*, Comptes Rendus Mécanique, **341**, 298–303, 2013.

55. A. CARPINTERI, P. CORNETTI, A. SAPORA, *Nonlocal elasticity: an approach based on fractional calculus*, Meccanica, **49**, 2551–2569, 2014.

56. Y. POVSTENKO, *Generalized theory of diffusive stresses associated with the time-fractional diffusion equation and nonlocal constitutive equations for the stress tensor*, Computers & Mathematics with Applications, **78**, 1819–1825, 2019.

57. V.E. TARASOV, E.C. AIFANTIS, *Non-standard extensions of gradient elasticity: fractional non-locality, memory and fractality*, Communications in Nonlinear Science and Numerical Simulation, **22**, 197–227, 2015.

58. Y. POVSTENKO, *Fractional nonlocal elasticity and solutions for straight screw and edge dislocations*, Physical Mesomechanics, **23**, 547–555, 2020.

59. R. GORENFLO, F. MAINARDI, *Fractional calculus and stable probability distributions*, Archives of Mechanics, **50**, 3, 377–388, 1998.

60. A.A. KILBAS, H.M. SRIVASTAVA, J.J. TRUJILLO, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

61. I. PODLUBNY, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

62. Y. POVSTENKO, *Straight disclinations in nonlocal elasticity*, International Journal of Engineering Science, **33**, 575–582, 1995.

63. R. GORENFLO, J. LOUTCHKO, YU. LUCHKO, *Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivatives*, Fractional Calculus and Applied Analysis, **5**, 491–518, 2002.

64. A.P. PRUDNIKOV, YU.A. BRYCHKOV, O.I. MARICHEV, *Integrals and Series, Vol. 1: Elementary Functions*, Gordon and Breach Science Publishers, Amsterdam, 1986.

65. A.P. PRUDNIKOV, YU.A. BRYCHKOV, O.I. MARICHEV, *Integrals and Series, Vol. 2: Special Functions*, Gordon and Breach Science Publishers, Amsterdam, 1986.

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